

SCHMIDT GAMES AND CONDITIONS ON RESONANT SETS

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ABSTRACT. Winning sets of Schmidt's game enjoy a remarkable rigidity. Therefore, this game (and modifications of it) have been applied to many examples of complete metric spaces (X, d) to show that the set of 'badly approximable points' $\mathbf{Bad}(\mathcal{F})$, with respect to a given family \mathcal{F} of resonant sets in X , is a winning set. For these examples, strategies were deduced that are, in most cases, strongly adapted to the specific dynamics and properties of the underlying setting. We introduce a new modification of Schmidt's game which is a combination of the ones of [18] and [20]. This modification allows us to axiomatize the conditions on the collection of resonant sets under which $\mathbf{Bad}(\mathcal{F})$ is a winning set. Moreover, we discuss properties of winning sets of this modification and verify our conditions for several examples - among them, the set $\mathbf{Bad}(\bar{r})$ of badly approximable vectors in \mathbb{R}^n , \mathbb{C}^n and \mathbb{Z}_p^2 with weights and the set of geodesic rays in proper geodesic CAT(-1) spaces which avoid a suitable collection of convex subsets.

1. INTRODUCTION AND MAIN RESULT

1.1. Introduction. We begin with a motivation. Let (X, d) be a metric space, μ a Borel probability measure and $T : X \rightarrow X$ a measure-preserving transformation. Let $A \subset X$ be a set of positive μ -measure. Then for μ -almost every point $x \in X$, the orbit $\mathcal{T}(x)$ of x hits A infinitely many times. The *shrinking target problem*, due to Hill and Velani [14], considers sets shrinking in time. More precisely, one considers a sequence of nested measurable sets $A_n \subset X$ and is interested in the properties of the points $x \in X$ whose orbit hit A_n for infinitely many times n . Such points are called *well approximable* in analogy with Diophantine approximation.

For instance, identify the projective real line $\mathbb{R} \cup \{\infty\}$ with the unit tangent space at a suitable point of the modular surface $\mathbb{H}^2/SL_2(\mathbb{Z})$. Then the well-approximable real numbers in the classical sense correspond to geodesics which enter a shrinking neighborhood of the only cusp of $\mathbb{H}^2/SL_2(\mathbb{Z})$ infinitely often. This is a set of full Lebesgue-measure. Conversely, a badly-approximable real number corresponds to a geodesic which avoids (i.e. does not enter) a certain neighborhood of the cusp. The set of badly-approximable numbers is of Lebesgue-measure zero, yet of full Hausdorff-dimension.

The following question has first been considered by Kristensen, Thorn and Velani [27] from a slightly different viewpoint. Given a countable index set Λ and a family of sets $\{R_\lambda \subset X : \lambda \in \Lambda\}$, called *resonant sets*, what kind of properties does the set admit, for which each of its point is not contained in certain (shrunked) neighborhoods of this family. More precisely, consider additionally a family of contractions $\{f_\lambda : \mathbb{R}^+ \rightarrow X : \lambda \in \Lambda\}$, where $R_\lambda \subset f_\lambda(t_1) \subset f_\lambda(t_2)$ for all $t_2 \leq t_1 < \infty$. Denote this family by

$$\mathcal{F} = (\Lambda, R_\lambda, f_\lambda).$$

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With respect to the family \mathcal{F} , we define the set of *badly approximable points* by

$$\mathbf{Bad}(\mathcal{F}) \equiv \{x \in X : \exists c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda \in \Lambda} f_\lambda(c)\}. \quad (1.1)$$

Kristensen, Thorn and Velani [27] already determined the Hausdorff-dimension of $\mathbf{Bad}(\mathcal{F})$ in a suitable framework using Ahlfors-regular measures.

In our paper, we want to use a different approach via modified Schmidt games. Winning sets of Schmidt's game (and modifications of it, called Schmidt games) enjoy a remarkable rigidity (see Subsection 2.1) which has been exploited by many authors. This can be seen from the list [1, 2, 4, 5, 6, 7, 9, 17, 18, 20, 24, 25, 26, 28, 31]. However, in most cases, strategies are deduced which are strongly adapted to the specific dynamics and properties of the considered example. The purpose of our paper is twofold. Firstly, we introduce a modification of Schmidt's game which is a combination of the one due to Kleinbock, Weiss [18] and McMullen [20] (as well as Broderick et al. [24]). This version of the game requires a weaker setting than Schmidt games and satisfies similar but weaker properties. Secondly, we state conditions on a given collection of resonant sets and the metric space X , under which there always exists a winning strategy for this modified game. The conditions concern mainly the (local) structure of the resonant sets in the space X and their distribution in X (both with respect to their "size").

1.2. Main Results. Although our main result, Theorem 2.4, will be stated in the setting of general metric spaces, we first illustrate it for the special case where $X = \mathbb{R}^n$ is the Euclidean space. In fact, we want to point out that for this case already Dani in [4, 5] deduced conditions on the resonant sets (as well as similar ones for the recently called \mathcal{L} -sets by Dani and Shah in [6]) under which $\mathbf{Bad}(\mathcal{F})$ is a winning set. Their conditions also concern for one, the (local) structure of the resonant sets, and for second, their distribution in \mathbb{R}^n ; for the precise statement see Theorem 3.2 in [4] and [6].

We now give a first version of our main result in its simplest form where we assume conditions related to the ones of [4, 6] (see Subsection 2.3). For a countable index set Λ , let $\{R_\lambda \subset \mathbb{R}^n : \lambda \in \Lambda\}$ be a collection of resonant sets, where to each R_λ we assign a *size* $s_\lambda \geq 0$ and the contraction $f_\lambda(c) \equiv \bigcup_{x \in R_\lambda} B(x, e^{-(s_\lambda+c)}) = \mathcal{N}_{e^{-(s_\lambda+c)}}(R_\lambda)$,¹ for $c \geq 0$. Suppose that the resonant sets are *nested* with respect to their sizes, that is, if $s_\lambda \leq s_\beta$ then $R_\lambda \subset R_\beta$, and, that the sizes $\{s_\lambda\} \subset \mathbb{R}^+$ are discrete. We say that the family $\mathcal{F} = (\Lambda, R_\lambda, f_\lambda)$ is *locally contained in metric spheres* if for all closed metric balls $B = B(x, e^{-r})$, and for all $\lambda \in \Lambda$ with $s_\lambda \leq r$ (and $B \cap R_\lambda$ possibly empty), there exists a metric sphere $S = \partial B(y_{B,\lambda}, e^{-r_{B,\lambda}})$ with $r_{B,\lambda} \in [-\infty, s_\lambda]$ such that $B \cap R_\lambda \subset S$. Note that we also allow $r_{B,\lambda} = -\infty$ but then assume that $y_{B,\lambda} \in \partial_\infty \mathbb{R}^n$ is a point at infinity so that we interpret S as an affine hyperplane. In particular, \mathcal{F} is locally contained in metric spheres if for any two distinct points $x \in R_\lambda, y \in R_\beta$ we have $d(x, y) > 2 \min\{e^{-s_\lambda}, e^{-s_\beta}\}$.

Theorem 1.1. *Let the family $\mathcal{F} = (\Lambda, R_\lambda, f_\lambda)$ be as above. If the resonant sets are nested, their sizes are discrete and \mathcal{F} is locally contained in metric spheres, then $\mathbf{Bad}(\mathcal{F})$ is a winning set.*

Note that the above theorem in particular applies to the set of badly approximable vectors $\mathbf{Bad}(\frac{1}{n}, \dots, \frac{1}{n})$ (Subsection 3.1) as well as to the set of endpoints in \mathbb{R}^n of lifts of bounded geodesic rays (starting at a fixed point) in a cusped finite-volume hyperbolic manifold

¹ Here, $B(x, r) \equiv \{y \in \mathbb{R}^n : d(x, y) \leq r\}$, $r > 0$, is the closed Euclidean ball around $x \in \mathbb{R}^n$ and $\mathcal{N}_\varepsilon(A)$ is the closed ε -neighborhood of a set $A \subset \mathbb{R}^n$.

(Subsection 3.6). In Section 3, these and further examples are discussed in more detail.

Outline of the paper. In Section 2, we first recall the ψ -modified Schmidt game due to [18] and its properties. We introduce our weaker version of the game in this setting and deduce weaker properties for this game. Moreover, we consider different conditions on the collection of resonant sets and on the metric space under which the set of badly approximable points is a winning set for the respective versions of the game. Finally, we discuss diffusion properties of the space X and on suitable (decaying) measures supported on X under which the deduced conditions on the resonant sets are satisfied.

In Section 3, we verify the conditions for the examples of the set of badly approximable vectors in \mathbb{R}^n , \mathbb{C}^n and \mathbb{Z}_p^2 with weights, of the ones with respect to a sequence of matrices and separated sets, as well as of the set of sequences in the Bernoulli-shift which avoid periodic sequences. Moreover, in more details, we consider the set of geodesics in a proper geodesic CAT(-1)-space which avoid certain convex subsets such as a collection of disjoint horoballs or neighborhoods of geodesic lines and orbit points.

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2. SCHMIDT GAMES

In this section, we combine two versions of Schmidt's game due to [18] and [20] in order to introduce a new modification. We first introduce the setting of this section which is the notion of [18]. Let (X, d) be a complete metric space. Fix $t_* \in \mathbb{R} \cup \{-\infty\}$ and define $\Omega \equiv X \times (t_*, \infty)$, the set of *formal balls* in X . Assume we are given a partial ordering \leq on Ω such that²

$$(x, t + s) \leq (x, t), \text{ for all } s \geq 0. \quad (2.1)$$

Let $\mathcal{C}(X)$ be the set of nonempty compact subsets of X . Assume that there exists a function $\psi : (\Omega, \leq) \rightarrow (\mathcal{C}(X), \subset)$ which is monotonic, that is,

$$\bar{\omega} \leq \omega \implies \psi(\bar{\omega}) \subset \psi(\omega). \quad (2.2)$$

For instance, if X is proper, set $t_* = -\infty$ and for $x \in X$, $r > 0$, let $B(x, r) \equiv \{y \in X : d(x, y) \leq r\} \in \mathcal{C}(X)$. The *standard pair* (\leq_s, ψ_s) is given by the partial ordering and monotonic function

$$(\bar{x}, \bar{t}) \leq_s (x, t) : \iff d(x, \bar{x}) + e^{-\bar{t}} \leq e^{-t}, \quad \psi_s(x, t) \equiv B(x, e^{-t}), \quad (2.3)$$

which satisfy (2.1) and (2.2). Note that if $(\bar{x}, \bar{t}) \leq_s (x, t)$ we have $\psi_s(\bar{x}, \bar{t}) \subset \psi_s(x, t)$ but the converse is not true in general.

² This assumption was not made by [18] but is essential for our purpose.

2.1. The ψ -modified Schmidt game. We recall the (ψ, a_*) -modified Schmidt game due to [18], where $a_* \geq 0$. Two players, A and B , pick numbers a and b both bigger than a_* . Player B starts with his first move by choosing a formal ball $\omega_1 = (x_1, t) \in \Omega$. Due to (2.1), player A can (and must) choose a formal ball $\bar{\omega}_1 = (\bar{x}_1, t_1 + a) \in \Omega$ such that $\bar{\omega}_1 \leq \omega_1$. Also player B continues by choosing a formal ball $\omega_2 = (x_2, t_1 + a + b) \in \Omega$ such that $\omega_2 \leq \bar{\omega}_1$. The game continues in this manner and we obtain a nested sequence of compact sets

$B_1 \equiv \psi(\omega_1) \supset A_1 \equiv \psi(\bar{\omega}_1) \supset B_2 \equiv \psi(\omega_2) \supset \dots \supset B_k \equiv \psi(\omega_k) \supset A_k \equiv \psi(\bar{\omega}_k) \supset \dots$, where $\omega_k = (x_k, t_k)$ and $\bar{\omega}_k = (\bar{x}_k, \bar{t}_k)$ satisfy

$$t_k = t_1 + (k-1)(a+b), \text{ and } \bar{t}_k = t_1 + (k-1)(a+b) + a.$$

The intersection of compact nested sets, given by

$$\bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} A_k,$$

is nonempty and compact. A subset $S \subset X$ is called (ψ, a_*, a, b) -winning, if player A can find a *strategy* which guarantees that $\bigcap_{k \geq 1} B_k$ intersects S , no matter what B 's choices are. The set S is called (ψ, a_*, a) -winning if S is (ψ, a_*, a, b) -winning for every $b > a_*$. S is (ψ, a_*) -winning if it is (ψ, a_*, a) -winning for some $a > a_*$ and ψ -winning if it is (ψ, a_*) -winning for some $a_* \geq 0$.

With respect to the standard partial ordering \leq_s and function ψ_s , the game described above coincides with the original (α, β) -Schmidt game for the choice

$$a = -\log(\alpha), \quad b = -\log(\beta), \quad a_* = 0, \quad t_* = -\infty. \quad (2.4)$$

If $X = \mathbb{R}^n$ is the Euclidean space and $(\leq, \psi) = (\leq_s, \psi_s)$, then winning sets enjoy the following properties (see [5, 28, 24]).

1. A winning set is dense and has Hausdorff-dimension n ,
2. a countable intersection of α -winning sets is α -winning,
3. winning sets are preserved by bi-Lipschitz homeomorphisms, and,
4. winning sets are incompressible.

Unfortunately, these properties are not satisfied in general; in fact, see [18], Proposition 5.2, for a ψ -winning set which is of Hausdorff-dimension zero in a space of positive dimension. However, the following (and further) properties for the ψ -modified Schmidt game can be found in [18].

1. Let $S_i \subset X$, $i \in \mathbb{N}$, be a sequence of (ψ, a_*, a) -winning sets. Then, $\bigcap_{i \geq 1} S_i$ is also (ψ, a_*, a) -winning.
2. Let $\Omega_i = X_i \times (t_*, \infty)$, and (\leq_i, ψ_i) be given for $i = 1, 2$. Suppose that $S_i \subset X_i$ is a (ψ_i, a_*) -winning set for $i = 1, 2$. Then $S_1 \times S_2$ is a $(\psi_1 \times \psi_2, a_*)$ -winning set in $X_1 \times X_2$ with the product metric, where $\psi_1 \times \psi_2(x_1, x_2, t) \equiv \psi_1(x_1, t) \times \psi_2(x_2, t)$.

Moreover, let μ be a locally finite Borel measure on X . Denote by $O(x, r) \equiv \{y \in X : d(x, y) < r\}$ the open metric ball around x . The *lower pointwise dimension* of μ at $x \in \text{supp}(\mu)$ is defined by

$$d_\mu(x) \equiv \liminf_{r \rightarrow 0} \frac{\log(\mu(O(x, r)))}{\log r}.$$

For every open $U \subset X$ with $\mu(U) > 0$,

$$d_\mu(U) \equiv \inf_{x \in U \cap \text{supp}(\mu)} d_\mu(x),$$

which is known to be a lower bound for the Hausdorff-dimension of $\text{supp}(\mu) \cap U$ (see [8], Proposition 4.9 (a)). The measure μ is called *Federer* if there are $K > 0$ and $R > 0$ such that for all $x \in \text{supp}(\mu)$ and $0 < r < R$,

$$\mu(O(x, 3r)) \leq K\mu(O(x, r)).$$

In the case that we consider the standard pair (\leq_s, ψ_s) , i.e., we focus on the classical Schmidt-game, the following lower estimate on the Hausdorff-dimension is given.

Proposition 2.1 ([18], Proposition 5.1). *If S is a winning set (in the sense of Schmidt) in a complete metric space X which supports a Federer measure μ with $X = \text{supp}(\mu)$, then for every nonempty open set $U \subset X$, we have $\dim(S \cap U) \geq d_\mu(U)$, where \dim stands for the Hausdorff-dimension.*

If μ satisfies a *power law*, that is, there exist δ, c_1, c_2 and $R > 0$ such that for every $0 < r < R$ and $x \in \text{supp}(\mu)$ we have

$$c_1 r^\delta \leq \mu(O(x, r)) \leq c_2 r^\delta,$$

then μ is Federer and we have $d_\mu(x) = \delta$.

2.2. The weak ψ -modified Schmidt game. For $b_* \geq 0$, consider the following modification of rules for the players A and B . Fix a parameter $b > b_*$. Player B starts again with a formal ball $\omega_1 = (x_1, t_1) \in \Omega$. Now, player A is allowed to choose a (possibly empty) set $A_1 \subset X$ under the condition that A leaves B a *legal* move, that is, B can (and must) choose a formal ball $\omega_2 = (x_2, t_1 + b) \in \Omega$ such that

$$\psi(\omega_2) \subset \psi(\omega_1) - A_1. \quad (2.5)$$

Note that (2.5) is always possible by (2.1) and (2.2), since A might choose Y_1 empty. The game continues in this manner and we obtain a nested sequence

$$B_1 \supset (B_1 - A_1) \supset B_2 \supset (B_2 - A_2) \supset B_3 \supset \cdots \supset B_k \supset (B_k - A_k) \supset \cdots,$$

where $B_k = \psi(x_k, t_k)$ with $t_k = t_1 + (k-1)b$ and $A_k \subset X$. If the nonempty compact set $\bigcap_{k \geq 1} B_k$ intersects a given set $S \subset X$, then A *wins* this game. The set S is called (ψ, b_*, b) -*weakly-winning* if player A finds a strategy such that A wins for every possible game, given the parameter b . S is called (ψ, b_*) -*weakly-winning* if it is (ψ, b_*, b) -*weakly-winning* for every $b > b_*$ and ψ -*weakly-winning* if it is (ψ, b_*) -*weakly-winning* for some $b_* \geq 0$.

Clearly, with respect to the standard partial ordering \leq_s and function ψ_s , the game described above is similar to the modification due to McMullen [20] (only defined on \mathbb{R}^n), called *absolute-winning game*, if we require all the sets $A_k = \psi_s(y_k, t_k + b)$ to be balls, $(y_k, t_k + b) \in \Omega$, and set

$$b = -\log(\beta), \quad b_* = \log(3), \quad t_* = -\infty.$$

Note that an absolute-winning set in \mathbb{R}^n is also winning.

The difference to the original ψ -game is that, rather than forcing B in a certain direction, A can block B 's choice in the next move. If we require for all sets $A_k \subset X$ which A chooses that there exists a formal ball $\bar{\omega} = (\bar{x}, t_k + b_*)$ such that, with ω_k the choice of B ,

$$\psi(\bar{\omega}) \subset \psi(\omega_k) - A_k, \quad (2.6)$$

then B always finds a legal move ω_{k+1} and a ψ -weakly-winning set is ψ -winning.

Lemma 2.2. *A (ψ, a_*, a) -winning set $S \subset X$ is (ψ, b_*) -weakly-winning for every $b_* \geq a_* + a$. Conversely, if (2.6) is satisfied, then a (ψ, b_*) -weakly-winning set S is (ψ, a_*) -winning for all $a_* \geq b_*$.*

Proof. First assume that (2.6) is satisfied. Given $a, b > a_* \geq b_*$, set $\bar{b} = a + b > b_*$. Let player A play the (ψ, a_*, a, b) -modified Schmidt game and consider a further player \bar{A} who plays the weak (ψ, b_*, \bar{b}) -modified Schmidt game. Suppose that player B has chosen his k -th move $\omega_k = (x_k, t_1 + (k-1)(a+b)) = (x_k, (k-1)\bar{b})$. By (2.6), \bar{A} chooses a set $A_k \subset X$ such that there exists a formal ball $\bar{\omega} = (\bar{x}, t_k + b_*) \leq \omega_k$ with

$$\psi(\bar{\omega}) \subset \psi(\omega_k) - A_k.$$

By (2.1), (2.2) and since $a > b_*$, there exists a formal ball $\bar{\omega}_{k+1} = (\bar{x}_{k+1}, t_k + a) = (\bar{x}_{k+1}, t_1 + (k-1)(a+b) + a) \leq \bar{\omega}$ which we take as A 's choice. Any move $\omega_{k+1} = (x_{k+1}, t_1 + k(a+b)) = (x_{k+1}, t_1 + k\bar{b}) \leq \bar{\omega}_{k+1}$ of B is a legal move in both games. Since \bar{A} has a (weakly-)winning strategy, we see that

$$\bigcap_{k \geq 1} \psi(\bar{\omega}_k) = \bigcap_{k \geq 1} \psi(\omega_k)$$

intersects S . Hence, A wins and S is a (ψ, a_*, a, b) -winning set.

Conversely, a winning strategy for the (ψ, a_*, a) -modified Schmidt game defines a winning strategy of the new version for $b_* = a + a_*$. In fact, let $b > a_*$ and $\bar{b} = b + a > b_*$. If player A chose, according to the strategy for the ψ -modified Schmidt game, the set $\psi(\bar{x}_k, t_k + a) \subset \psi(x_k, t_k)$ then player \bar{A} , who plays the (ψ, b_*, \bar{b}) -game, removes the set $\psi(\bar{x}_k, t_k + a)^C$. Any choice $\omega_{k+1} = (x_{k+1}, t_k + a + b)$ with $\psi(\omega_{k+1}) \subset \psi(x_k, t_k)$ of B is a legal move with respect to both games and we proceed in this way. Since S is (ψ, a_*, a) -winning we have $\bigcap_{k \geq 1} \psi(\bar{\omega}_k) \cap S \neq \emptyset$. Hence, S is also (ψ, b_*, b) -weakly-winning. \square

Hence, in view of the properties of ψ -winning sets (see Subsection 2.1), we will consider conditions which ensure that (2.6) is satisfied so that the weak ψ -modified Schmidt game is at least as strong as the ψ -modified Schmidt game. However, some of the properties of ψ -winning sets can still be true in the weaker setting.

In fact, let S be a (ψ, b_*, b) -weakly-winning set. In order to estimate the lower bound for the Hausdorff-dimension of S , we consider the conditions given by [18] and only need to modify $(\mu 2)$ below:

(MSG1) For any open set $\emptyset \neq U \subset X$ there is $\omega \in \Omega$ such that $\psi(\omega) \subset U$.

(MSG2) There exist $C, \sigma > 0$ such that $\text{diam}(\psi(x, t)) \leq Ce^{-\sigma t}$ for all $(x, t) \in \Omega$.

Note that if (MSG1) is satisfied, a ψ -weakly-winning set is dense. Let moreover μ be a locally finite Borel measure on X such that:

($\mu 1$) For every formal ball $\omega \in \Omega$ we have $\mu(\psi(\omega)) > 0$.

($\mu 2$) There exists $r_* \in \mathbb{R}$ and a constant $c = c(b) > 0$ with the following property: If $\omega_k \in \Omega$ with $\text{diam}(\psi(\omega_k)) < e^{-r_*}$ is a choice of B in the (ψ, b_*, b) -game, there exist legal moves $\omega_{k+1}^1, \dots, \omega_{k+1}^n \leq \omega_k$ of B with respect to the choice of A according to the (ψ, b_*, b) -winning strategy, which are essentially disjoint³ and such that

$$\mu\left(\bigcup_{i=1 \dots n} \psi(\omega_{k+1}^i)\right) \geq c \cdot \mu(\psi(\omega_k)). \quad (2.7)$$

Note that from (MSG1) and ($\mu 1$), μ must have full support, i.e. $\text{supp}(\mu) = X$.

³ That is, $\mu(\psi(\omega_{k+1}^i) \cap \psi(\omega_{k+1}^j)) = 0$ when $i \neq j$.

Proposition 2.3. *Suppose that $X, \Omega, (\leq, \psi)$ and the measure μ satisfy (MSG1-2) and (μ 1-2) with respect to a (ψ, b_*, b) -weakly-winning set S . Then for every nonempty open set $U \subset X$ we have that*

$$\dim(S \cap U) \geq d_\mu(U) + \frac{\log(c(b))}{\sigma b},$$

where σ and $c = c(b)$ are the constants of (MSG2) and (μ 2).

Proof. Similarly to the proof of [18], Theorem 2.7, one constructs a strongly treelike countable family of compact subsets of X whose limit set $A_\infty \cap U$ is a subset of $S \cap U$. The difference is that, instead of using the choices of A , we use the choices of B given in (μ 2) in order to obtain that

$$\dim(A_\infty \cap U) \geq d_\mu(U) + \frac{\log(c)}{\sigma b}.$$

Hence, the proof follows. \square

2.3. The framework, conditions on the resonant sets and a strategy. Let \bar{X} be a metric space and X a subset of \bar{X} which is, with the induced metric, a complete metric space. In many applications, we are interested in playing the ψ -game on X but do not require the resonant sets to be contained in X but in \bar{X} . Therefore, let $\bar{\Omega} = \bar{X} \times (t_*, \infty)$ and $\Omega = X \times (t_*, \infty) \subset \bar{\Omega}$. Let $(\bar{\leq}, \bar{\psi})$ on $\bar{\Omega}$ satisfy (2.1) and (2.2), which induces the pair (ψ, \leq) on Ω , defined by

$$\psi(\omega) \equiv \bar{\psi}(\omega) \cap X, \quad \omega \in \Omega,$$

which also satisfies (2.1) and (2.2).⁴ For a subset $Y \subset \bar{X}$ and $t > t_*$, we call $(Y, t) \equiv \{(y, t) : y \in Y\}$ *formal neighborhood*, and define $\mathcal{P}(\bar{X})$ to be the set of formal neighborhoods. Define the $\bar{\psi}$ -neighborhood of $(Y, t) \in \mathcal{P}(\bar{X})$ by

$$\bar{\psi}(Y, t) \equiv \bigcup_{y \in Y} \bar{\psi}(y, t).$$

Note that by (2.1) and (2.2), $\bar{\psi}(Y, t + s) \subset \bar{\psi}(Y, t)$ for all $s \geq 0$. Moreover, ψ is called d_* -contracting for $d_* > 0$, if for all $x \in X$ and $(y, t) \in \Omega$, we have

$$x \in \psi(y, t + d_*) \Rightarrow \psi(x, t + d_*) \subset \psi(y, t). \quad (2.8)$$

Clearly, (\leq_s, ψ_s) is d_* -contracting for all $d_* \geq \log(2)$.

Now, let Λ be a countable index set and $\{R_\lambda \subset \bar{X} : \lambda \in \Lambda\}$ be a family of *resonant sets*, where to every R_λ is assigned a *size* $s_\lambda \geq s_*$ with $t_* < s_* \in \mathbb{R}$. We consider the contractions of the $(\bar{\psi}, s_\lambda)$ -neighborhoods of R_λ ,

$$f_\lambda(s) \equiv \bar{\psi}(R_\lambda, s_\lambda + s) \subset \bar{\psi}(R_\lambda, s_\lambda), \quad s \geq 0.$$

Denote this family by

$$\mathcal{F} = (\bar{X}, X, \Lambda, R_\lambda, s_\lambda, \bar{\leq}, \bar{\psi}), \quad (2.9)$$

or simply by $\mathcal{F} = (\Lambda, R_\lambda, s_\lambda)$ if there is no confusion about the spaces under consideration. We then define the set of *badly approximable points* by

$$\mathbf{Bad}(\mathcal{F}) = \{x \in X : \exists c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda \in \Lambda} f_\lambda(c)\}.$$

Assume that the family \mathcal{F} satisfies the following conditions.

⁴ It is also possible to consider two different pairs $(\bar{\leq}, \bar{\psi})$ on $\bar{\Omega}$ and (\leq, ψ) on Ω where ψ is not induced by $\bar{\psi}$.

(N) The resonant sets $\{R_\lambda\}$ are *nested* with respect to their sizes, that is, for $\lambda, \beta \in \Lambda$ we have

$$s_\lambda \leq s_\beta \implies R_\lambda \subset R_\beta. \quad (2.10)$$

(D) The sizes $\{s_\lambda\}$ are *discrete*, that is, for all $t > t_*$ we have

$$|\{\lambda \in \Lambda : s_\lambda \leq t\}| < \infty. \quad (2.11)$$

If \mathcal{F} is nested and discrete, then for every $r > t_*$, we let $\Lambda_r = \{\lambda \in \Lambda : s_\lambda \leq r\}$, which is at most finite. If Λ_r is nonempty, let $\lambda_r \in \Lambda_r$ such that $s_{\lambda_r} = \max\{s_\lambda : \lambda \in \Lambda_r\}$ and set $R(r) \equiv R_{\lambda_r}$, otherwise let $R(r)$ be empty. By (2.10), we have that $R_\lambda \subset R(r)$ for all $\lambda \in \Lambda_r$. Moreover, for $r > t_*$ and $b > 0$, we let

$$R(r, b) \equiv R(r) - R(r - b) \quad (2.12)$$

be the set of resonant points for which the 'minimal size' belongs to the spectrum $(r - b, r]$. We consider two conditions, a strong and a weak one, on the space X and the family \mathcal{F} .

(b_*^s) X is *strongly b_* -diffuse with respect to the family \mathcal{F}* for some $b_* \geq 0$, if there exists $t_* < r_* \in \mathbb{R}$ and $n \in \mathbb{N}$ such that, for all formal balls $\omega = (x, r) \in \Omega$ with $r > r_*$, there exists a formal ball $\omega' = (x', r + b_*) \in \Omega$ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(R_r, r + nb_*). \quad (2.13)$$

(b_*, d_*) X is *b_* -diffuse with respect to the family \mathcal{F}* for some $b_* \geq d_*$,⁵ if there exists $t_* < r_* \in \mathbb{R}$ and if for all $b > b_*$ there exists a $n = n(b) \in \mathbb{N}$ such that, for all formal balls $\omega = (x, r) \in \Omega$ with $r > r_*$, there exists a formal ball $\omega' = (x', r + (b - d_*)) \in \Omega$ ⁶ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(R(r, b), r + nb). \quad (2.14)$$

Note that, up to adding the same constant to all sizes s_λ , we can (and in fact always will) assume in the following that $s_* = 0 > r_*$. Under these conditions we have the following.

Theorem 2.4. *Let \mathcal{F} be a nested and discrete family. If X is strongly b_* -diffuse with respect to \mathcal{F} , then the set $\mathbf{Bad}(\mathcal{F})$ is (ψ, a_*) -winning for every $a_* \geq b_*$. If X is b_* -diffuse with respect to \mathcal{F} and ψ is d_* -contracting, then the set $\mathbf{Bad}(\mathcal{F})$ is (ψ, b_*) -weakly-winning.*

Condition (b_*^s) is too strong in general (see Subsection 3.3 and 3.6, Case 3.) but implies Condition ($b_* + d_*, d_*$) and is sufficient to guarantee that if $\mathbf{Bad}(\mathcal{F})$ is (ψ, b_*) -weakly-winning it is also (ψ, b_*) -winning by Lemma 2.2.

Proof of Theorem 2.4. We first show that $\mathbf{Bad}(\mathcal{F})$ is (ψ, b_*) -weakly-winning under the assumption that X is b_* -diffuse with respect to \mathcal{F} . For a parameter $b > b_*$ we need to find a winning-strategy for player A . Assume B chose the formal ball $\omega_1 = (x_1, t_1) \in \Omega$. Let $m \in \mathbb{N}$ such that $\bar{b} \equiv mb \geq t_1 + d_*$ and let $n = n(\bar{b})$ be as in (2.14). For $l \in \mathbb{N}_0$ let

$$R_l \equiv R(t_{ml+1} + d_*, \bar{b}) = R(t_{ml+1} + d_*) - R(t_{ml+1} + d_* - \bar{b}).$$

Recall that $t_k = t_1 + (k - 1)b$. For $k \geq 1$, assume that B chose the formal ball $\omega_k = (x_k, t_k) \in \Omega$. Let $k = ml + s$ with $l, s \in \mathbb{N}_0$, $1 \leq s \leq m$. By (2.14), there exists a formal ball $\omega'_l = (x'_l, t_{ml+1} + \bar{b}) \in \Omega$ such that

$$\psi(\omega'_l) \subset \psi(x_{ml+1}, t_{ml+1} + d_*) - \bar{\psi}(R_l, t_{ml+1} + d_* + n\bar{b}). \quad (2.15)$$

⁵ Here and in the following, d_* is the constant from Condition (2.8).

⁶ Alternatively, we can require that there exists a formal ball $\omega' = (x', r + b) \in \Omega$ with $x' \in (x, r + d_*)$ and satisfying (2.14). However, we decided to choose this version.

Hence, define player A 's strategy by choosing the complement in X

$$A_k \equiv \psi(x'_l, t_{ml+1} + sb)^C \cap X = \psi(x'_l, t_{ml+s} + b)^C \cap X. \quad (2.16)$$

Note that, since $x'_l \in \psi(x_{ml+1}, t_{ml+1} + d_*)$ and $b \geq d_*$, we have $\psi(x'_l, t_{ml+1} + b) \subset \psi(x_{ml+1}, t_{ml+1})$ by (2.8) as well as by (2.1) and (2.2). Moreover, $\psi(x'_l, t_{ml+s} + b) \subset \psi(x'_l, t_{ml+s})$ for $1 < s \leq m$. Thus, B is left with exactly one legal move ω_{k+1} as in (2.5), namely $\omega_{k+1} = (x'_l, t_{ml+s} + b)$ and we proceed with ω_{k+1} as above. In particular, $\omega_{m(l+1)+1} = (x'_l, t_{m(l+1)+1}) = (x'_l, t_{ml+1} + \bar{b}) = \omega'_l$.

Hence, let $x_0 \in \cap_{k \geq 1} \psi(\omega_k)$. Assume that $x_0 \in \bar{\psi}(R_{\lambda_0}, s_{\lambda_0})$ for some $\lambda_0 \in \Lambda$ (if no such λ_0 exists, then A has already won). Since $\bar{b} \geq t_1 + d_*$, we know that R_{λ_0} is covered by $R_{\lambda_0} \subset \cup_{l=0}^N R_l$ (where we let N be the minimal such integer). Thus, there exists $l \in \mathbb{N}_0$ such that $x_0 \in \bar{\psi}(R_l, s_{\lambda_0})$. If $l \geq 1$, then $t_{ml+1} + d_* - \bar{b} < s_{\lambda_0}$ from the definition of R_l , and by (2.15) we have that

$$\begin{aligned} x_0 &\notin \bar{\psi}(R_l, t_{ml+1} + d_* + n\bar{b}) = \bar{\psi}(R_l, t_{ml+1} + d_* - \bar{b} + (n+1)\bar{b}) \\ &\supset \bar{\psi}(R_l, s_{\lambda_0} + (n+1)\bar{b}), \end{aligned} \quad (2.17)$$

by (2.1) and (2.2). If $l = 0$, note that $t_1 + d_* > s_*$ and similarly,

$$\begin{aligned} x_0 &\notin \bar{\psi}(R_0, t_1 + d_* + n\bar{b}) \supset \bar{\psi}(R_0, s_{\lambda_0} + (t_1 + d_* + n\bar{b} - s_{\lambda_0})) \\ &\supset \bar{\psi}(R_0, s_{\lambda_0} + (t_1 + d_* + n\bar{b} - s_*)) \supset \bar{\psi}(R_0, s_{\lambda_0} + (n+1)\bar{b}), \end{aligned}$$

since $s_* < s_{\lambda_0}$ and $\bar{b} \geq t_1 + d_* - s_*$. This shows that

$$x_0 \notin \cup_{l=1}^N \psi(R_l, s_{\lambda_0} + (n+1)\bar{b}) \supset \bar{\psi}(R_{\lambda_0}, s_{\lambda_0} + (n+1)\bar{b}).$$

Therefore, $x_0 \in \mathbf{Bad}(\mathcal{F})$, since

$$x_0 \notin \bigcup_{\lambda \in \Lambda} \bar{\psi}(R_\lambda, s_\lambda + (n+1)\bar{b}).$$

Hence, A wins and we defined a winning strategy for the parameter $b > b_*$.

Now, let X be strongly b_* -diffuse with respect to \mathcal{F} . For $b > b_*$ and $n = n(b_*)$, we choose $A_k = \bar{\psi}(R_k, t_k + nb_*) \cap X$ instead of (2.16) where now $R_k \equiv R(t_k, b)$ for $k \geq 2$ and $R_1 \equiv R(t_1)$. For every $b > b_*$, the set R_k is contained in $R(t_k)$ and, by (2.13), there exists a formal ball $\omega' = (x', t_k + b_*) \in \Omega$ such that

$$\begin{aligned} \psi(x', t_k + b) &\subset \psi(\omega') \subset \psi(\omega_k) - \bar{\psi}(R(t_k), r + nb_*) \\ &\subset \psi(\omega_k) - \psi(R(t_k, b), t_k + nb_*) = \psi(\omega_k) - A_k, \end{aligned}$$

by (2.1) and (2.2). Hence, $(x', t_k + b)$ is a legal move for player B and we see similarly as in (2.17) that the strategy defined is (ψ, b_*, b) -weakly-winning. In particular, (2.6) is satisfied and the first claim of the Theorem follows from Lemma 2.2. \square

Remark. Assume that $X = \mathbb{R}^n$ is the Euclidean space and we are given a nested and discrete family $\mathcal{F} = \{\Lambda, R_\lambda, s_\lambda, \leq_s, \psi_s\}$ of resonant sets as in (2.9). Our conditions are neither weaker nor stronger than the ones required in [4, 5, 6]. For instance, if \mathcal{F} is locally contained in affine hyperplanes (in the sense of Subsection 1.2), then $\mathbf{Bad}(\mathcal{F})$ is a \mathcal{L} -set as defined in [6], Definition 3.1. Conversely, if $\mathbf{Bad}(\mathcal{F})$ is an \mathcal{L} -set and we have in Definition 3.1, [6] (in their notation) that uniformly in $\lambda \in (0, 1)$ it holds $P = \Lambda$, $S(p, t) = \mathcal{N}_{e^{-(s_p+t)}}(R_p)$, $\tau_p = s_p$ for all $p \in P$ and ρ is constant then Condition (b_*) is satisfied for $b_* > 0$ sufficiently large.

We want to consider a further condition, weaker than (b_*^s) but stronger than (b_*) , which is preserved under maps which satisfy some kind of bi-Lipschitz-property and by finite intersections. For this purpose, we need a further parameter.

(b_*, d_*, n_*) X is (b_*, d_*, n_*) -diffuse with respect to the family \mathcal{F} for some $b_* \geq d_*$ and $n_* \in \mathbb{N}$, if there exists $t_* < r_* \in \mathbb{R}$ and for all $b > b_*$ there exists a $n = n(b) \in \mathbb{N}$ such that, for all formal balls $\omega = (x, r) \in \Omega$ with $r > r_*$, there exists a formal ball $\omega' = (x', r + (b - d_*)) \in \Omega$ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(R(r, n_*b), r + nb). \quad (2.18)$$

Clearly, (b_*^s) implies $(b_* + d_*, d_*, n_*)$ for every $n_* \in \mathbb{N}$ and $d_* \geq 0$ which in turn implies $(b_* + d_*, d_*)$. First, let $F : (\bar{X}, \bar{\Omega}_{\bar{X}}, \leq_{\bar{X}}, \psi_{\bar{X}}) \rightarrow (\bar{Y}, \bar{\Omega}_{\bar{Y}}, \leq_{\bar{Y}}, \psi_{\bar{Y}})$ be a bijective map for which there exists $L_* \geq 0$ such that, for all formal balls $(x, r) \in \bar{\Omega}_{\bar{X}}$, we have

$$\psi_{\bar{Y}}(F(x), r + 2L_*) \subset F(\psi_{\bar{X}}(x, r + L_*)) \subset \psi_{\bar{Y}}(F(x), r). \quad (2.19)$$

If both $\psi_{\bar{X}} = \psi_s^{\bar{X}}$ and $\psi_{\bar{Y}} = \psi_s^{\bar{Y}}$, then F is a L_* -bi-Lipschitz map. Given a nested, discrete family of resonant sets $\mathcal{F}_X = (\bar{X}, X, \Lambda, R_\lambda, s_\lambda, \leq_{\bar{X}}, \psi_{\bar{X}})$, consider the induced family in \bar{Y} with $Y \equiv F(X)$,

$$\mathcal{F}_Y \equiv F(\mathcal{F}_X) \equiv (\bar{Y}, Y, \Lambda, F(R_\lambda), s_\lambda - L_*, \leq_{\bar{Y}}, \psi_{\bar{Y}}),$$

which is also nested and discrete. It is readily checked that $F(\mathbf{Bad}(\mathcal{F}_X)) = \mathbf{Bad}(\mathcal{F}_Y)$.

Proposition 2.5. *Let $F : (\bar{X}, \bar{\Omega}_{\bar{X}}, \leq_{\bar{X}}, \psi_{\bar{X}}) \rightarrow (\bar{Y}, \bar{\Omega}_{\bar{Y}}, \leq_{\bar{Y}}, \psi_{\bar{Y}})$ satisfy (2.19). If X is (strongly b_* -diffuse) (b_*, d_*, n_*) -diffuse with respect to \mathcal{F}_X with $2L_* \leq d_*$, then Y is (strongly $(b_* + 2L_*)$ -diffuse) $(b_*, 2d_* - 2L_*, n_*)$ -diffuse with respect to \mathcal{F}_Y .*

Proof. We first show that ψ_Y satisfies (2.8) for $\bar{d}_* = 2d_* - 2L_* \geq d_*$. In fact, let $(\bar{y}, t + d_* - 2L_*) \in \Omega_Y$ and $y \in \psi_Y(\bar{y}, t + \bar{d}) \subset \psi_Y(\bar{y}, t + d_*)$. From (2.19) we have

$$F^{-1}(y) \in F^{-1}(\psi_Y(\bar{y}, t + d_*)) \subset \psi_X(F^{-1}(\bar{y}), t + d_* - L_*).$$

By (2.8), we get $\psi_X(F^{-1}(y), t - L_* + d_*) \subset \psi_X(F^{-1}(\bar{y}), t - L_*)$, and again by (2.19),

$$\psi_Y(y, t - 2L_* + d_*) \subset F(\psi_X(F^{-1}(y), t - L_*)) \subset F(\psi_X(F^{-1}(\bar{y}), t - L_*)) \subset \psi_Y(\bar{y}, t).$$

Assume that X is (b_*, d_*, n_*) -diffuse with respect to \mathcal{F}_X . Let $(y, r) \in \Omega_Y$. For $b > b_*$, there exists $n \in \mathbb{N}$ and $\omega' = (\bar{x}, r + L_* + (b - d_*)) \in \Omega_X$ such that

$$\psi_X(\omega') \subset \psi_X(F^{-1}(y), r + L_*) - \psi_{\bar{X}}(R_{\bar{X}}(r + L_*, n_*b), r + L_* + nb). \quad (2.20)$$

From (2.19) we have

$$\psi_Y(F(\bar{x}), r + (b - \bar{d})) \subset F(\psi_X(\bar{x}, r + L_* + (b - d_*))) \subset F(\psi_X(F^{-1}(y), r + L_*)) \subset \psi_Y(y, r).$$

Since $F(R_X(r + L_*, t)) = R_Y(r, t)$, we have for $m \in \mathbb{N}$ such that $mb \geq nb + 2L_*$,

$$\begin{aligned} \psi_Y(R_Y(r, n_*b), r + mb) &\subset \psi_{\bar{Y}}(R_{\bar{Y}}(r, n_*b), r + 2L_* + nb) \\ &\subset F(\psi_{\bar{X}}(R_{\bar{X}}(r + L_*, n_*b), r + L_* + nb)). \end{aligned}$$

By (2.20) we know that $F(\psi_X(\omega'))$ is disjoint to $F(\psi_{\bar{X}}(R_{\bar{X}}(r + L_*, n_*b), r + L_* + nb_*))$ and hence we see that Y is $(b_*, 2d_* - 2L_*, n_*)$ -diffuse with respect to \mathcal{F}_Y .

The case when X is strongly b_* -diffuse with respect to \mathcal{F}_X follows similarly. \square

Now consider finitely many families $\mathcal{F}_i = (\Lambda^i, R_{\lambda^i}^i, s_{\lambda^i}^i)$, $i = 1, \dots, n_*$, of nested and discrete families in \bar{X} . When X is strongly b_* -diffuse with respect to each \mathcal{F}_i , we know from Theorem 2.4 and properties of ψ -modified Schmidt games that $\cap_{i=1}^{n_*} \mathbf{Bad}(\mathcal{F}_i)$ is (ψ, b_*) -winning (and the same is true for countable intersections). In the weaker setting, we know the following.

Proposition 2.6. *If X is (b_*, d_*, n_*) -diffuse with respect to each family \mathcal{F}_i and ψ is d_* -contracting, then $\cap_{i=1}^{n_*} \mathbf{Bad}(\mathcal{F}_i)$ is (ψ, b_*) -weakly-winning.*

Proof. Assume that X is (b_*, d_*, n_*) -diffuse with respect to each family \mathcal{F}_i and let $b > b_*$. We only need to modify the strategy for player A in (2.16). In fact, if $\omega_1 = (x_1, t_1) \in \Omega$ is the first move of B , we let $\bar{b} = mb \geq t_1 + d_*$ and $\bar{m} = n_* m$. Let $k = l\bar{m} + im + s$ for $l, i, s \in \mathbb{N}_0$, $0 \leq i < n_*$ and $1 \leq s \leq m$. Denote by $R_l^i = R^{i+1}(t_{l\bar{m}+im+1} + d_*, n_* \bar{b})$, where R^{i+1} is the subset of the resonant sets with respect to \mathcal{F}_{i+1} . Hence, there exists a formal ball $\omega_l^i = (x_l^i, t_{l\bar{m}+im+1} + \bar{b}) \in \Omega$ such that

$$\psi(\omega_l^i) \subset \psi(x_{l\bar{m}+im+1}, t_{l\bar{m}+im+1} + d_*) - \bar{\psi}(R_l^i, r + d_* + n\bar{b}).$$

We therefore define

$$A_k = \psi(x_l^i, t_{l\bar{m}+im+1} + sb)^C \cap X = \psi(x_l^i, t_k + b)^C \cap X.$$

As in Theorem 2.4 we see that B is left with a legal move and proceed with ω_{k+1} .

Thus, for $i = 0, \dots, n_* - 1$ and $x \in \cap_{l \geq 1} \psi(\omega_{l\bar{m}+im+1}) = \cap_{j \geq 1} \psi(\omega_j)$, we deduce as in (2.17) that $x \in \mathbf{Bad}(\mathcal{F}_{i+1})$. In particular, $x \in \cap_i \mathbf{Bad}(\mathcal{F}_{i+1})$ which is thus a (ψ, b_*) -weakly-winning set. \square

Assume for $i = 1, \dots, n_*$ that $\mathcal{F}_i = (\bar{Y}_i, Y_i, \Lambda^i, R_{\lambda^i}^i, s_{\lambda^i}^i, \leq^i, \psi^i)$ is a nested discrete family in \bar{Y}_i and that $F_i : \bar{Y}_i \rightarrow \bar{X}$ is a bijective map satisfying (2.19) for some constant L_*^i with $F(Y_i) = X$. As a corollary, if each Y_i is (b_*, d_*, n_*) -diffuse with respect to \mathcal{F}_i and $2d_*^i - 2L_*^i = d_*$, where ψ_X is d_* -contracting for d_* , then

$$\cap_{i=1}^{n_*} F_i(\mathbf{Bad}(\mathcal{F}_i)) \subset X$$

is a (ψ_X, b_*) -weakly-winning set. This is a weaker version of the property that winning sets for Schmidt's game are incompressible.

Remark. Let $\Omega_i = \bar{X}_i \times (t_*, \infty)$, and $(\leq_i, \bar{\psi}_i)$ be given for $i = 1, 2$, where $\bar{\psi}_1 \times \bar{\psi}_2(x_1, x_2, t) = \bar{\psi}_1(x_1, t) \times \bar{\psi}_2(x_2, t)$. Moreover, let $\mathcal{F}_i = (\bar{X}_i, X_i, \Lambda, R_{\lambda}^i, s_{\lambda}, \leq_i, \bar{\psi}_i)$ be nested and discrete with the same index set and the same sizes. If (b_*^s) , (b_*, d_*) or (b_*, d_*, n_*) respectively is satisfied for both X_i and \mathcal{F}_i , then (b_*^s) , (b_*, d_*) or (b_*, d_*, n_*) respectively is satisfied for $X_1 \times X_2$ with respect to $\mathcal{F} = (\bar{X}_1 \times \bar{X}_2, X_1 \times X_2, \Lambda, R_{\lambda}^1 \times R_{\lambda}^2, s_{\lambda}, \bar{\psi}_1 \times \bar{\psi}_2)$.

2.4. Diffuse spaces and decaying measures. While Conditions (N) and (D) are satisfied for many examples, one has to check Condition (b_*^s) or (b_*, d_*, n_*) in general. We will now discuss under which conditions on the space $X \subset \bar{X}$ it only suffices to assume that the resonant sets are nicely structured and distributed.

In the following, let $(\leq, \bar{\psi})$ be the standard pair (\leq_s, B) . We give a special class of diffuse spaces X in which the resonant sets might be more general than points but are still nicely structured and distributed with respect to (\leq_s, B) . More precisely, let \mathcal{S} be the set of metric spheres $S(\bar{\omega}) \equiv \{y \in \bar{X} : d(\bar{x}, y) = e^{-t}\}$, where $\bar{\omega} = (\bar{x}, t) \in \bar{\Omega}$. For $b_* \geq 0$ we call $X \subset \bar{X}$ b_* -diffuse with respect to \mathcal{S} , if there exists $r_* \in \mathbb{R} \cup \{-\infty\}$ such that for any formal ball $\omega = (x, r) \in \Omega$ and metric sphere $S(\bar{\omega}) = S(\bar{x}, t)$ with $r \geq t \geq r_*$ there exists a formal ball $\omega' = (x', r + b_*) \in \Omega$ such that

$$B(x', e^{-(r+b_*)}) \subset B(x, e^{-r}) - \mathcal{N}_{e^{-(r+b_*)}}(S(\bar{\omega})).$$

Consider a nested and discrete family $\mathcal{F} = (\Lambda, R_{\lambda}, s_{\lambda})$ of resonant sets in \bar{X} , satisfying the condition that \mathcal{F} is *locally contained in metric spheres*, that is, for every R_{λ} , $\lambda \in \Lambda$, and for all formal balls $\bar{\omega} = (\bar{x}, r) \in \bar{\Omega}$ with $r \geq s_{\lambda}$, there exists a metric sphere $S(\omega_{\lambda, \bar{\omega}}) \in \mathcal{S}$ such that $R_{\lambda} \cap B(\bar{x}, 2 \cdot e^{-r}) \subset S(\omega_{\lambda, \bar{\omega}})$ and $\omega_{\lambda, \bar{\omega}} = (x_{\lambda, \bar{\omega}}, r_{\lambda, \bar{\omega}}) \in \bar{\Omega}$ with $r \geq r_{\lambda, \bar{\omega}}$. If

moreover X is b_* -diffuse with respect to \mathcal{S} , $b_* > 0$, then for any formal ball $\omega = (x, r) \in \Omega$ there exists a formal ball $\omega' = (x', r + b_*) \in \Omega$ such that

$$B(x', e^{-(r+b_*)}) \subset B(x, e^{-r}) - \mathcal{N}_{e^{-(r+b_*)}}(S(\omega_{\lambda, \omega})) \subset B(x, e^{-r}) - \mathcal{N}_{e^{-(r+b_*)}}(R_\lambda). \quad (2.21)$$

Hence, we see that X is strongly b_* -diffuse with respect to the family \mathcal{F} .

Remark. We can consider a more general family and pair than \mathcal{S} and $(\bar{\leq}_s, \bar{\psi}_s)$.

Let $\bar{X} = \mathbb{R}^n$ be the Euclidean space. Since affine hyperplanes can be seen as metric spheres at infinity, our definition above is a generalization of the one given by [24]: A closed subset $X \subset \mathbb{R}^n$ is called k -dimensionally b_* -diffuse ($0 \leq k < n$, $b_* > 0$) if there exists a $r_* \in \mathbb{R} \cup \{-\infty\}$ such that for all $r > r_*$, $x \in X$, and for any k -dimensional affine subspace $L \subset \mathbb{R}^n$, there exists $x' \in X$ such that

$$B(x', e^{-(r+b_*)}) \subset B(x, e^{-r}) - \mathcal{N}_{e^{-(r+b_*)}}(L).$$

As a special case, when $k = 0$ and \bar{X} is a metric space, $X \subset \bar{X}$ is called β -diffuse in [19] if there exists $r_* \in \mathbb{R} \cup \{-\infty\}$ such that for any formal ball $(x, r) \in \Omega$ with $r \geq r_*$, and for all formal balls $(\bar{x}, r + \beta_0) \in \bar{\Omega}$ there exists $(x', r + \beta) \in \Omega$ such that

$$B(x', e^{-(r+\beta)}) \subset B(x, e^{-r}) - B(\bar{x}, e^{-(r+\beta)}).$$

For a class of β -diffuse spaces, let X be a *uniformly perfect* metric space, that is, there exists $r_* \in \mathbb{R} \cup \{-\infty\}$ and a constant $0 < \nu < \infty$ such that for any metric ball $B(x, e^{-r})$, $x \in X$, $r > r_*$ with $X - B(x, e^{-r}) \neq \emptyset$, we have

$$(B(x, e^{-r}) - B(x, e^{-(\nu+r)})) \cap X \neq \emptyset.$$

Similar to [19], Lemma 2.4, we show the following.

Lemma 2.7. *If X is uniformly perfect with respect to $\nu > 0$, then X is β -diffuse for any $\beta \geq \nu + \log(4) + \log(3/4)$.*

Proof. Let $x \in X$, $r > r_*$ and $\bar{x} \in X$. If $d(x, \bar{x}) > 2e^{-(r+\beta)}$ then for $x' = x$ we have $B(x', e^{-(r+\beta_0)}) \subset B(x, e^{-r}) - B(\bar{x}, e^{-(r+\beta)})$. On the other hand, if $d(x, \bar{x}) \leq 2e^{-(r+\beta)}$ then $B(\bar{x}, e^{-(r+\beta_0)}) \subset B(x, 3e^{-(r+\beta_0)})$. Let $c = \beta - \nu - \log(4) \geq \log(3/4)$. Since X is uniformly perfect, there exists $x' \in (B(x, e^{-(r+c)}) - B(x, e^{-(\nu+r+c)})) \cap X$. Hence,

$$4e^{-(r+\beta)} \leq e^{-(r+\nu+c)} < d(x, x') \leq e^{-(r+c)} \leq \frac{3}{4}e^{-r} \leq e^{-r} - e^{-(r+\beta)}.$$

Again we have $B(x', e^{-(\beta+r)}) \subset B(x, e^{-r}) - B(\bar{x}, e^{-(\beta+r)})$. \square

Consider the following examples of b_* -diffuse and uniformly perfect spaces $X \subset \bar{X}$.

1. Let $n \geq 1$ and $b_* > \log(3)$. The unit sphere $X = S^n \subset \mathbb{R}^{n+1} = \bar{X}$ is k -dimensionally b_* -diffuse for every $k \leq n$. If $X = \bar{X} = S^n$, then X is b_* -diffuse.
2. If Γ is a non-elementary finitely generated Kleinian group acting on the hyperbolic space \mathbb{H}^{n+1} (the unit ball model), then the limit set $X = \Lambda\Gamma \subset S^n = \bar{X}$ of Γ is uniformly perfect by [15]. For the definitions see Subsection 3.6.
3. Let $n \geq 1$. If $\Sigma^+ = \{0, \dots, n\}^{\mathbb{N}}$ denotes the set of one-sided sequences in the symbols $\{0, 1, \dots, n\}$, together with the metric $d^+(w, \bar{w}) \equiv e^{-\min\{i \geq 1: w(i) \neq \bar{w}(i)\}}$ for $w \neq \bar{w}$ and $d(w, w) \equiv 0$, then (Σ^+, d) is compact and b_* -diffuse for $b_* = 1$.
4. Let T be a tree of valence at least 3 with the path metric such that every edge is of length 1. For a vertex point $o \in T$, let d_o be the visual metric (see Section 3 for the definition) on the set ∂T of ends of T . Then $(\partial T, d_o)$ is compact and b_* -diffuse for $b_* = 1$.

5. If X is the support of a locally finite Borel measure on $\bar{X} = \mathbb{R}^n$ which is δ -decaying, then there exists $b_* = b_*(\delta) > 0$ such that X is $(n - 1)$ -dimensionally b_* -diffuse. For the definition and the proof see below. Moreover, the following result is due to [16]. Let $\{S_1, \dots, S_k\}$ be an irreducible family of contracting self-similarity maps of \mathbb{R}^n satisfying the open set condition and let X be the attractor. If μ is the restriction of the δ -dimensional Hausdorff-measure to X , $\delta = \dim(X)$, then μ is δ -decaying and satisfies a power law with respect to the exponent δ . Particular examples of such sets are regular Cantor-sets, Koch's curve and the Sierpinski gasket.

Remark. Note that for the Bernoulli-shift Σ^+ metric spheres are a finite union of cylinder sets; hence of Hausdorff-dimension $\log(n) = \dim(\Sigma^+)$. Therefore, diffuseness might not be a condition on the dimension of the resonant sets.

Now, let $\mathcal{F} = (\Lambda, R_\lambda, s_\lambda)$ be a nested and discrete family of resonant sets. The following proposition is a corollary of Theorem 2.4 and a simple criterion.

Proposition 2.8. *Let X be a β -diffuse complete metric space and \mathcal{F} as above. Assume there exists $c > 0$ such that for any two distinct points $x \in R_\lambda$ and $y \in R_{\lambda'}$ we have*

$$d(x, y) > c \cdot \min\{e^{-s_\lambda}, e^{-s_{\lambda'}}\}. \quad (2.22)$$

Then (2.13) is satisfied, that is, X is strongly b_ -diffuse with respect to the family \mathcal{F} , where $b_* = \beta$. Hence, $\mathbf{Bad}(\mathcal{F})$ is absolute winning (in the sense of McMullen).*

Proof. Note first that, up to adding $\bar{c} = \log(c) - \log(3e^{-b_*} + 1)$ to every size s_λ , we may assume that $d(x, y) > (3e^{-b_*} + 1) \min\{e^{-s_\lambda}, e^{-s_{\lambda'}}\}$ in (2.22). Let $B = B(x, e^{-r})$ be any closed metric ball with center $x \in X$. Let R_λ be a resonant set with $r \geq s_\lambda > r_*$. Assume there is a point $\bar{x} \in R_\lambda$ such that $B(\bar{x}, e^{-(r+b_*)})$ intersects $B(x, e^{-(r+b_*)})$. In particular, we then have $d(x, \bar{x}) \leq 2e^{-(r+b_*)}$. Let $\bar{y} \in R_\lambda$ be distinct from \bar{x} . Then, by (2.22) we have

$$d(x, \bar{y}) \geq d(\bar{y}, \bar{x}) - d(x, \bar{x}) > (3e^{-b_*} + 1)e^{-r} - 2e^{-(r+b_*)} = e^{-r} + e^{-(r+b_*)},$$

which shows that $B(\bar{y}, e^{-(r+b_*)})$ does not intersect B . Since X is β -diffuse, there exists a ball $B(x', e^{-(r+b_*)}) \subset B - B(\bar{x}, e^{-(r+b_*)}) = B - \bigcup_{y \in R_\lambda} B(y, e^{-(r+b_*)})$ with $x' \in X$. If no such point $\bar{x} \in R_\lambda$ exists, then we choose $x' = x$ and $B(x', e^{-(r+b_*)})$. Thus, X is strongly b_* -diffuse with respect to the family \mathcal{F} . \square

Remark. Note that Condition (2.22) is similar to, but in fact weaker than the condition

$$d(x, y) \geq \sqrt{e^{-s_\lambda} e^{-s_{\lambda'}}}.$$

For $X = \mathbb{R}^n$, this condition was considered in a similar setting by [5] and recently by [6] where it was called \mathcal{B} -set.

As a further tool to show that a space satisfies (b_*^s) , (b_*, d_*) or (b_*, n_*, c_*) with respect to a given family, we use the notion of decaying measures on X , introduced in [16]. Recall that a locally finite Borel measure μ on \mathbb{R}^n is called δ -decaying, for some constants c , $\delta > 0$, if there exists $r_* > 0$ such that for all balls $B = B(x, e^{-r})$ with $x \in \text{supp}(\mu)$, $r > r_*$ and for all affine hyperplanes $L \subset \mathbb{R}^n$, we have

$$\mu(B \cap \mathcal{N}_{e^{-(r+s)}}(L)) \leq ce^{-\delta s} \mu(B), \quad s \in \mathbb{R}. \quad (2.23)$$

The function $f(s) = ce^{-\delta s}$ determines the rate of the decay of the measure of $\mathcal{N}_{e^{-(r+s)}}(L)$ in B in terms of the relative size s of this neighborhood of L . Note moreover that it is readily checked that $d_\mu(x) \geq \delta$ for $x \in \text{supp}(\mu)$.

We want to extend this notion with respect to a function $\bar{\psi}$ defined on $(\bar{\Omega}, \bar{\leq})$. Given a subset $S \subset \bar{X}$, we call it $\bar{\psi}$ -Borel, if $\bar{\psi}(S, t)$ is a Borel set for all $t \in \mathbb{R}$. Let X be the support of a locally finite Borel measure μ and assume that all $\{x\} \subset X$ are $\bar{\psi}$ -Borel. Moreover, let $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be a function, non-decreasing in the first and non-increasing in the second argument, where we denote $f_b(\cdot) \equiv f(b, \cdot)$. In what follows we let $\mathcal{F} = (\lambda, R_\lambda, s_\lambda)$ be a nested and discrete family. If every resonant set R_λ is $\bar{\psi}$ -Borel,⁷ we call the family \mathcal{F} *measurable* and consider the following conditions.

(μ^s) μ is called *strongly $(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F}* for some $b_* \geq 0$ and a function f as above, if there exists a $r_* \in \mathbb{R}$ such that for all formal balls $\omega = (x, r) \in \Omega$ with $r > r_*$ and for all $s \geq b_*$ we have

$$\mu(\psi(x, r) \cap \bar{\psi}(R(r), r + s)) \leq f_{b_*}(s) \mu(\psi(x, r)).$$

(μ) μ is called *$(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F}* for some $b_* \geq 0$ and a function f as above, if there exists a $r_* \in \mathbb{R}$ such that for all formal balls $\omega = (x, r) \in \Omega$ with $r > r_*$, for all $b \geq b_*$ and $s \geq b_*$ we have

$$\mu(\psi(x, r) \cap \bar{\psi}(R(r, b), r + s)) \leq f_b(s) \mu(\psi(x, r)). \quad (2.24)$$

Again, the function f determines the rate of decay of the measure of the neighborhood of the resonant set in $\psi(x, r)$. For constants $n_* \in \mathbb{N}$, and $0 \leq d_* \leq b_*$ we say that f is (b_*, d_*, n_*) -decaying if there exists a constant $c < 1$ such that

$$f(n_*b + d_*, b - 2d_*) \leq c \quad \text{for all } b > b_* + 2d_*, \quad (2.25)$$

and strongly (b_*, d_*) -decaying if $f_{b_*}(b_* - 2d_*) \leq c$. Note that if μ is strongly $(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F} , we consider it $(\bar{\psi}, f, b_*)$ -decaying with respect to the function $f(b, s) = f_{b_*}(s)$, independent on b , which is clearly (b_*, d_*, n_*) -decaying for every $n_* \in \mathbb{N}$.

Consider moreover that $\bar{\psi}$ is d_* -contracting with respect to \mathcal{F} , that is, there exists a constant $d_* > 0$ such that ψ is d_* -contracting and such that for all formal neighborhoods $(Y, t) = (R_\lambda, t) \in \mathcal{P}(\bar{X})$, $\lambda \in \Lambda$, or formal balls $(Y, t) = (y, t) \in \Omega$ and for all $x \in X$,

$$x \notin \bar{\psi}(Y, t) \implies \psi(x, t + d_*) \cap \bar{\psi}(Y, t + d_*) = \emptyset. \quad (2.26)$$

Proposition 2.9. *Let $\bar{\psi}$ be d_* -contracting with respect to \mathcal{F} and μ be a locally finite Borel measure with $X = \text{supp}(\mu)$. If μ is (strongly) $(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F} and a function f which is $((b_*, d_*)$ -decaying) (b_*, d_*, n_*) -decaying. Then X is (strongly \bar{b}_* -diffuse) (\bar{b}_*, d_*, n_*) -diffuse with respect to \mathcal{F} for $\bar{b}_* = b_* + 2d_*$.*

Proof. Assume that μ is $(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F} and f is (b_*, d_*, n_*) -decaying. For $\bar{b}_* = b_* + 2d_*$ and $b > \bar{b}_*$ note that $R(r, n_*b) \subset R(r + d_*, n_*b + d_*)$ and $b - d_* \geq b_*$. Let $\omega = (x, r) \in \Omega$ with $r > r_*$. We have

$$\begin{aligned} & \mu(\psi(x, r + d_*) \cap \bar{\psi}(R(r, n_*b), r + b - 2d_*)) \\ & \leq \mu(\psi(x, r + d_*) \cap \bar{\psi}(R(r + d_*, n_*b + d_*), r + b - 2d_*)) \\ & \leq f(n_*b + d_*, b - 2d_*) \mu(\psi(x, r + d_*)). \end{aligned}$$

Since for $b > \bar{b}_* = b_* + 2d_*$ we have $f(n_*b + d_*, b - 2d_*) \leq c < 1$ by (2.25), there exists a point $\bar{x} \in \psi(x, r + d_*) \cap \bar{\psi}(R(r, n_*b), r + b - 2d_*)^c$. By (2.8) and since $b - d_* \geq b_*$, we have for $\omega' = (\bar{x}, r + (b - d_*)) \in \Omega$ that $\psi(\omega') \subset \psi(\bar{x}, r + d_*) \subset \psi(x, r)$. Furthermore, (2.26) implies that $\psi(\bar{x}, r + (b - d_*))$ is disjoint from $\bar{\psi}(R(r, n_*b), r + b - d_*)$. This shows that X is (\bar{b}_*, d_*, n_*) -diffuse with respect to \mathcal{F} .

The case when μ is strongly $(\bar{\psi}, f, b_*)$ -decaying follows similarly. \square

⁷ In this case, also $R(r, b) = R(r) - R(r - b)$ is $\bar{\psi}$ -Borel for every $r \in \mathbb{R}$, $b > 0$.

We say that μ satisfies a power law with respect to ψ and the exponent $\tau > 0$, if there exist r_* , $c_1, c_2 > 0$ such that for all $\omega = (x, t) \in \Omega$ with $t > r_*$,

$$c_1 e^{-\tau t} \leq \mu(\psi(x, t)) \leq c_2 e^{-\tau t}.$$

Theorem 2.10. *Let $\bar{\psi}$ be d_* -contracting and μ be a locally finite Borel measure with $X = \text{supp}(\mu)$ satisfying a power law with respect to the exponent τ . Assume that either μ is $(\bar{\psi}, f, b_*)$ -decaying where f is (b_*, d_*, n_*) -decaying or X is strongly b_* -diffuse with respect \mathcal{F} . If moreover (MSG1-2) are satisfied, then for all open sets $\emptyset \neq U \subset X$ we have*

$$\dim(S \cap U) \geq d_\mu(U).$$

Proof. Let first μ be $(\bar{\psi}, f, b_*)$ -decaying with respect to \mathcal{F} and f be (b_*, d_*, n_*) -decaying. Note that clearly, $(\mu 1)$ is satisfied. Let $b > \bar{b}_* = b_* + 2d_*$ and $\omega_1 = (x_1, t_1) \in \Omega$ be the first move of B such that, by (MSG1), $\psi(\omega_1) \subset U$. Let again $m \in \mathbb{N}$ such that $\bar{b} = mb \geq t_1 + 2d_*$. For $k = lm + s \geq 1$, $1 \leq s \leq m$, such that $\text{diam}(\psi(\omega_k)) < e^{-r_*}$ as in $(\mu 2)$, let $x = x_{lm+1}$ and $t = t_{lm+1}$. As in the proof of Proposition 2.9, let $x^1 \in \psi(x, t + d_*) \cap \bar{\psi}(R(t + d_*, \bar{b}), t + \bar{b} - 2d_*)^C$. We moreover see that

$$\begin{aligned} & \mu(\psi(x, t + d_*) \cap (\psi(x^1, t + \bar{b} - d_*) \cup \bar{\psi}(R(t + d_*, \bar{b}), t + \bar{b} - d_*))) \\ & \leq c_2 e^{-\tau(t + \bar{b} - d_*)} + f(\bar{b}, \bar{b} - d_*) \cdot \mu(\psi(x, t + d_*)) \\ & \leq \left(\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \bar{b}} + c\right) \mu(\psi(x, t + d_*)). \end{aligned}$$

Since $c < 1$, for \bar{b} sufficiently large such that $\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \bar{b}} + c < 1$, there exists a point

$$x^2 \in \psi(x_{lm+1}, t_{lm+1} + d_*) \cap \bar{\psi}(R(t + d_*, \bar{b}), t + \bar{b} - d_*)^C \cap \psi(x^1, t + \bar{b} - d_*)^C.$$

With the same arguments as above, we have that $\psi(x^2, r + \bar{b})$ is contained in $\psi(x, r)$ and disjoint from both, $\psi(x^1, t + \bar{b})$ and $\bar{\psi}(R(t + d_*, \bar{b}), t + \bar{b})$. Iterating this argument until $(N+1)\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \bar{b}} - c > 1$, we obtain N points x^1, \dots, x^N such that $\psi(x^i, t + \bar{b}) \subset \psi(x, t)$, $i = 1, \dots, N$, are disjoint and also disjoint to $\bar{\psi}(R(t + d_*, \bar{b}), t + \bar{b})$. Moreover, we have

$$\begin{aligned} \mu\left(\bigcup_{i=1}^N \psi(x^i, t + \bar{b})\right) & \geq N c_1 e^{-\tau(t + \bar{b})} \geq \frac{(N+1)c_1}{2} e^{-\tau(t + \bar{b})} \geq \frac{(1-c)c_1^2 e^{-2\tau d_*}}{2c_2} e^{\tau \bar{b}} e^{-\tau(t + \bar{b})} \\ & \geq \frac{(1-c)c_1^2 e^{-2\tau d_*}}{2c_2^2} \mu(\psi(x, t)) \equiv c_0 \cdot \mu(\psi(x, t)), \end{aligned}$$

Furthermore, each of the $\omega_{l(m+1)+1}^i = (x^i, t_{l(m+1)+1}) = (x^i, t + \bar{b}) \leq (x, t) = \omega_{lm+1}$ is a formal ball, which was chosen according to the (ψ, \bar{b}_*, b) -winning-strategy of A ; compare with (2.16). Instead, we can view these formal balls as legal moves $\omega_{l+2}^i \equiv \omega_{l(m+1)+1}^i \leq (x, t) = \omega_{l+1}$ of B with respect to a winning strategy of A of the $(\psi, \bar{b}_*, \bar{b})$ -game. Hence, we see that $(\mu 2)$ is satisfied for the parameter \bar{b} with $c = c(\bar{b}) \geq c_0$. Finally, Proposition 2.3 implies that

$$\dim(S \cap U) \geq d_\mu(U) + \frac{\log(c_0)}{\sigma m b}, \quad (2.27)$$

and the proof follows since (2.27) is true for every $b > \bar{b}_*$.

If X is strongly b_* -diffuse with respect to \mathcal{F} , there exists $\psi(\bar{x}, t + b_*) \subset \psi(\omega) - \bar{\psi}(R(t), t + nb_*)$. With similar arguments, we can choose disjoint formal balls $\psi(x_i, t + b)$, $i = 1, \dots, N$, contained in $\psi(\bar{x}, t + b_*)$, where N is such that $(N+1)(\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau b}) + c > 1$ and each of the formal balls is a legal move of B according to the (ψ, b_*, b) -winning strategy of A . The proof then follows similarly. \square

3. APPLICATIONS

In order to discuss our conditions, we consider several examples. Some of the results are already known and we either simplify their proofs, weaken the assumptions to our weaker setting or improve them.

Given a complete metric space X in \bar{X} with a pair $(\bar{\leq}, \bar{\psi})$ on $\bar{\Omega} = \bar{X} \times (t_*, \infty)$ satisfying (2.1) and (2.2), we are left with defining a suitable nested discrete family of resonant sets \mathcal{F} . We want to set the focus on the distribution of the resonant sets by verifying the Conditions (b_*^s) or (b_*, d_*, n_*) respectively as well as finding suitable measures for the purpose of determining the Hausdorff-dimension of $\mathbf{Bad}(\mathcal{F})$.

3.1. $\mathbf{Bad}_{\mathbb{R}^n}(\bar{r})$. For $n \geq 1$, let $\bar{r} \in \mathbb{R}^n$ with $r^1, \dots, r^n \geq 0$ such that $\sum r^i = 1$. Let $\mathbf{Bad}_{\mathbb{R}^n}(\bar{r})$ be the set of points $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1, \dots, n} |qx_i - p_i|^{1/r^i} \geq c(\bar{x})/q,$$

for every $q \in \mathbb{N}$ and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$. The set $\mathbf{Bad}_{\mathbb{R}^1}(1)$ is the classical set of badly approximable numbers and the set $\mathbf{Bad}_{\mathbb{R}^n}(1/n, \dots, 1/n)$ agrees with the set of badly approximable vectors.

As in [18], we let $D_0 = [-1, 1]^n$ and consider the contraction $\Phi_t^{\bar{r}}(D_0)$, which is the box

$$[-e^{-(1+r^1)t}, e^{-(1+r^1)t}] \times \dots \times [-e^{-(1+r^n)t}, e^{-(1+r^n)t}].$$

Let $\bar{\Omega} = \mathbb{R}^n \times (0, \infty)$. Define $\bar{\psi} : \bar{\Omega} \rightarrow \mathbb{R}^n$ by

$$\bar{\psi}(\bar{x}, t) \equiv \bar{x} + \phi_t^{\bar{r}}(D_0),$$

which induces a partial ordering on $\bar{\Omega}$,

$$(\bar{x}, t) \bar{\leq}_{\bar{\psi}} (\bar{y}, r) : \iff \bar{\psi}(\bar{x}, t) \subset \bar{\psi}(\bar{y}, r). \quad (3.1)$$

Then the pair $(\bar{\leq}_{\bar{\psi}}, \bar{\psi})$ satisfies (2.1) and (2.2). We improve a result of [9, 18, 27].

Theorem 3.1. *Let X be the support of a locally finite Borel measure which is δ -decaying with respect to $\bar{\psi}$.⁸ Then, for any map $F : (\mathbb{R}^n, \bar{\leq}_{\bar{\psi}}, \bar{\psi}) \rightarrow (\bar{Y}, \bar{\leq}_{\bar{Y}}, \bar{\psi}_{\bar{Y}})$ satisfying (2.19), $F(\mathbf{Bad}_{\mathbb{R}^n}(\bar{r}) \cap X)$ is ψ_Y -winning in $F(X)$.*

Proof. For $k \in \Lambda \equiv \mathbb{N}_{\geq 2}$ we define the set of rational vectors

$$R_k \equiv \{\bar{p}/q : \bar{p} \in \mathbb{Z}^n, 0 < q < k\}$$

as resonant set and define its size by $s_k \equiv \log(k) + \log(2^{2n}) + \log(n!)$. The family $\mathcal{F} = (\mathbb{R}^n, X, \mathbb{N}_{\geq 2}, R_k, s_k, \bar{\leq}_{\bar{\psi}}, \bar{\psi})$ is nested and discrete and we show that X is strongly b_* -diffuse with respect to \mathcal{F} . Choose any resonant set R_k and let $\omega = (x, r) \in \Omega$ be a formal ball such that $s_k < r$. Note that $\psi(\omega)$ is contained in the box $\psi(x, r - \log(2))$ for which the sidelights ρ_i satisfy

$$\rho_1 \dots \rho_n = 2e^{-(1+r^1)(r-\log(2))} \dots 2e^{-(1+r^n)(r-\log(2))} < 2^{2n} e^{-(1+n)s_k} \leq \frac{1}{n!k^{n+1}}.$$

We use the following version of the 'Simplex Lemma' due to Davenport and Schmidt where the version of this lemma can be found in [27], Lemma 4.

Lemma 3.2. *Let $D \subset \mathbb{R}^n$ be a box of side lengths ρ_1, \dots, ρ_n such that $\rho_1 \dots \rho_n < 1/(n!k^{n+1})$. Then there exists an affine hyperplane L such that $R_k \cap D \subset L$.*

⁸ That is, there is $\delta > 0$ and $\bar{c} > 0$ such that for any affine hyperplane $L \subset \mathbb{R}^n$ and for any formal ball $\omega = (x, r) \in \Omega$, we have $\mu(\psi(\omega) \cap \bar{\psi}(L, r + s)) \leq \bar{c}e^{-\delta s} \mu(\psi(\omega))$, $s \geq 0$.

Hence, for $b_* \geq \log(2)$, we conclude that $\psi(\omega) - \bar{\psi}(R(r), r + b_*) = \psi(\omega) - \bar{\psi}(L, r + b_*)$. Since μ is δ -decaying with respect to $\bar{\psi}$, it follows that μ is strongly (ψ, f, b_*) -decaying with respect to \mathcal{F} and the function $f_{b_*}(s) = \bar{c}e^{-\delta s}$. Clearly, if $b_* = b_*(c, \delta) \geq \log(2)$ is sufficiently large, then $f_{b_*}(b_* - 2\log(2)) \leq c < 1$ and f is strongly $(b_*, \log(2))$ -decaying. Moreover, $\bar{\psi}$ is $\log(2)$ -contracting with respect to \mathcal{F} so that X is strongly b_* -diffuse with respect to \mathcal{F} by Proposition 2.9. By Proposition 2.5, $Y = F(X)$ is strongly $(b_* + 2L_*)$ -diffuse with respect to the nested, discrete family $F(\mathcal{F})$. Theorem 2.4 shows that $\mathbf{Bad}(F(\mathcal{F}))$ is ψ_Y -winning.

Finally, if $\bar{x} \in \mathbf{Bad}(\mathcal{F})$, there exists a constant $c = c(\bar{x}) < \infty$ such that for all \bar{p}/q , where $\bar{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$, $\bar{x} \notin \bar{\psi}(R_{q+1}, s_{q+1} + c) \supset \bar{\psi}(\bar{p}/q, s_{q+1} + c)$. Hence, for some $i \in \{1, \dots, n\}$, we have

$$|x_i - p_i/q| \geq e^{-(1+r^i)(s_{q+1}+c)} \geq \frac{e^{-(1+\max\{r^i\})c}}{2^{2n+2n!}} q^{-(1+r^i)},$$

and we see that $\mathbf{Bad}(\mathcal{F}) \subset \mathbf{Bad}_{\mathbb{R}^n}(\bar{r}) \cap X$. Since a set containing a ψ_Y -(weakly)-winning set is ψ_Y -(weakly)-winning, $\mathbf{Bad}(F(\mathcal{F})) = F(\mathbf{Bad}(\mathcal{F})) \subset F(\mathbf{Bad}_{\mathbb{R}^n}(\bar{r}) \cap X)$ is ψ_Y -winning. \square

Now let μ be the Lebesgue-measure on \mathbb{R}^n . It follows from [16], Lemma 9.1, that μ is δ -decaying with respect to $\bar{\psi}$ for $\delta = 1 + \min\{r^1, \dots, r^n\}$. Note also that Conditions (MSG1-2) are satisfied. Moreover, μ satisfies a power law with respect to ψ and the exponent $n + 1$. In fact, for all $(x, t) \in \Omega$ we have $\mu(\psi(x, t)) = 2^n e^{-(n+1)t}$. By Theorem 2.10 we get

$$n \geq \dim(\mathbf{Bad}(\mathcal{F})) \geq d_\mu(\mathbb{R}^n) = n.$$

Remark. Assume that $X = Y = \mathbb{R}^n$ and F is as above. If moreover $\psi_Y = \psi$, then $F(\mathbf{Bad}_{\mathbb{R}^n}(\bar{r})) \cap \mathbf{Bad}_{\mathbb{R}^n}(\bar{r})$ is ψ -winning and the same is true for countable intersections of such sets. As pointed out in [18], a class of examples of F satisfying (2.19) with $Y = \mathbb{R}^n$, $\psi_Y = \psi$, is given by nonsingular affine maps for which the linear part commutes with $\phi_1^{\bar{r}}$.

3.2. $\mathbf{Bad}_{\mathbb{Z}_p^n}(\bar{r})$. Let p be a prime number, $|\cdot|_p$ the p -adic absolute value and \mathbb{Z}_p be the p -adic integers in the p -adic field \mathbb{Q}_p . For $n \geq 1$, let again $\bar{r} \in \mathbb{R}^n$ with $r^1, \dots, r^n \geq 0$ such that $\sum r^i = 1$. Because of the different properties of the p -adic field, we need to adjust the definition of badly approximable p -adic vectors. For further details, we refer to [27]. Let $\mathbf{Bad}_{\mathbb{Z}_p^n}(\bar{r})$ be the set of points $\bar{x} = (x_1, \dots, x_n) \in \mathbb{Z}_p^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1, \dots, n} |x_i - \frac{z_i}{q}|_p^{1/(1+r^i)} \geq c(\bar{x}) \max\{|z_1|, \dots, |z_n|, |q|\}^{-1},$$

for all $(z_1, \dots, z_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$. Let $d(x, y) \equiv |x - y|_p$ be the p -adic metric on \mathbb{Z}_p . For $(\bar{x}, t) \in \mathbb{Q}_p^n \times (0, \infty)$ consider the box

$$\bar{\psi}(\bar{x}, t) \equiv B(x_1, e^{-(1+r^1)t}) \times \dots \times B(x_n, e^{-(1+r^n)t}).$$

Again, $(\bar{\psi}, \bar{\psi})$, with $\bar{\psi}$ as in (3.1), satisfies (2.1) and (2.2). For $n = 2$, it was already shown by [27] that (a slightly different version of) $\mathbf{Bad}_{\mathbb{Z}_p^2}(\bar{r})$ is of Hausdorff-dimension 2. We show the following stronger result.

Theorem 3.3. *$\mathbf{Bad}_{\mathbb{Z}_p^2}(\bar{r})$ is ψ -winning and $\dim(\mathbf{Bad}_{\mathbb{Z}_p^2}(\bar{r})) = 2$.*

Sketch of the proof. We first remark that implicitly in the proof of [27], Subsection 5.4, the following analogue of the Simplex Lemma is shown.

Lemma 3.4. *There exists $r_* > 0$ and $c_* > 0$ such that, if $\tilde{R}_t \equiv \{(z_1/q, z_2/q) \in \mathbb{Q}^2 : \max\{|z_1|, |z_2|, |q|\} \leq e^{t-c_*}\}$, then $\tilde{R}_t \cap \psi(\bar{x}, t)$ is contained in a p -adic line L for any $(\bar{x}, t) \in \Omega$ with $t > r_*$.*

We therefore let $\Lambda \equiv \mathbb{N}_{[r_*]}$ and define for $n \in \Lambda$ the resonant set

$$R_n \equiv \{(z_1/q, z_2/q) \in \mathbb{Q}_p^2 : z_1, z_2 \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } \max\{|z_1|, |z_2|, |q|\} \leq n\}$$

with the size $s_n \equiv \log(n) + \log(4) + c_*$, where c_* is as above. We show that $X = \mathbb{Z}_p^2$ is strongly b_* -diffuse with respect to the nested and discrete family $\mathcal{F} = (\mathbb{Q}_p^2, \mathbb{Z}_p^2, \mathbb{N}, R_n, s_n, \leq_{\bar{\psi}}, \bar{\psi})$. In fact, let $\omega = (\bar{x}, t) \in \Omega$ with $r_* < s_n \leq t$. Then $n \leq e^{t-c_*}$ and from Lemma 3.4 it follows that $R_n \cap \psi(\omega)$ is contained in a p -adic line L . As in the case of Theorem 3.1, for $b_* \geq \log(2)$, we have $\psi(\omega) - \bar{\psi}(R(t), t + b_*) = \psi(\omega) - \bar{\psi}(L, t + b_*)$. Moreover, as shown in [27], for $b_* > 0$ sufficiently large, a geometric argument implies that any number of disjoint boxes $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, intersecting L is bounded above by $C \cdot e^{b_*(1+\max\{r^1, r^2\})}$, where C is independent of b_* . On the other hand, we let $m \equiv \mu \times \mu$, where μ is the normalized Haar-measure on \mathbb{Q}_p . Hence, $\mu(\mathbb{Z}_p) = 1$ and $m(B(x_1, r_1) \times B(x_2, r_2)) = p^{-(t_1+t_2)}$ for $p^{-t_i} \leq r_i \leq p^{-t_i+1}$ and $t_i \in \mathbb{N}$, $i = 1, 2$. In particular, $p^{-4}e^{-3t} \leq m(\psi(\omega)) \leq e^{-3t}$ so that μ satisfies a power law with respect to ψ . Hence, for $b_* > 0$ sufficiently large, there exists a collection of disjoint boxes $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, whose number exceeds the one of its boxes intersecting L (independently from t). If we take such a box $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, not intersecting L , then $\psi(\bar{x}_i, t + 2b_*)$ is disjoint from $\psi(L, t + 2b_*)$ (if b_* is sufficiently large) and we see that X is strongly $2b_*$ -diffuse with respect to \mathcal{F} . Hence, $\mathbf{Bad}(\mathcal{F})$ is ψ -winning by Theorem 2.4. Moreover, by Theorem 2.10, it is of Hausdorff-dimension 2.

Finally, let $\bar{x} \in \mathbf{Bad}(\mathcal{F})$ and $(z_1/q, z_2/q) \in \mathbb{Q}^2$ with $\max\{|z_1|, |z_2|, |q|\} = n$. There exists $c(x) < \infty$ such that $\bar{x} \notin \psi(R_n, s_n + c(x)) \supset \psi((z_1/q, z_2/q), s_n + c(x))$. Hence, for some $i \in \{1, 2\}$ we have

$$|x_i - z_i/q|_p > e^{-(1+r^i)(s_n+c(x))} \geq e^{-(1+\max\{r^1, r^2\})(c_*+\log(4)+c(x))} n^{-(1+r^i)}.$$

Therefore, $\mathbf{Bad}(\mathcal{F}) \subset \mathbf{Bad}_{\mathbb{Z}_p^2}(\bar{r})$ which finishes the proof of the Theorem. \square

Remark. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers in \mathbb{C} . Let $\mathbf{Bad}_{\mathbb{C}^n}(\bar{r})$ be the set of points $\bar{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1, \dots, n} |qx_i - z_i|^{1/r^i} \geq c(\bar{x}) \cdot |q|^{-1},$$

for every $z_1, \dots, z_n, q \in \mathbb{Z}[i]$, $q \neq 0$. For $k \in \Lambda \equiv \mathbb{N}$ define the resonant set $R_k \equiv \{(z_1/q, \dots, z_n/q) \in \mathbb{C}^n : z_1, \dots, z_n, q \in \mathbb{Z}[i], 0 < |q| \leq k\}$ with size $s_k \equiv \log(k) + \log(2^n)$. Define ψ similarly as in Subsection 3.1 and 3.2. By showing the corresponding version of the Simplex Lemma in this setting (see [27], Theorem 18) it follows in the same fashion as above that $\mathbf{Bad}((\mathbb{C}^n, \mathbb{C}^n, \mathbb{N}, R_k, s_k, \leq_{\bar{\psi}}, \bar{\psi})) \subset \mathbf{Bad}_{\mathbb{C}^n}(\bar{r})$ is ψ -winning and of full Hausdorff-dimension.

3.3. $E(\mathcal{M}, \mathcal{Z})$. For the motivation, further generalizations and consequences of the following result, we refer to [25]. For $n \in \mathbb{N}$, let $\mathcal{M} = (M_k)$ be a sequence of real matrices $M_k \in GL(n, \mathbb{R})$ and $\mathcal{Z} = (Z_k)$ be a sequence of τ_k -separated⁹ subsets of \mathbb{R}^n . Define

$$E(\mathcal{M}, \mathcal{Z}) \equiv \{x \in \mathbb{R}^n : \exists c = c(x) > 0 \text{ such that } d(M_k x, Z_k) \geq c \cdot \tau_k \text{ for all } k \in \mathbb{N}\},$$

⁹ That is, for every $y_1, y_2 \in Z_k$ we have $d(y_1, y_2) \geq \tau_k > 0$.

where d is the Euclidean distance. The sequence \mathcal{M} is *lacunary*, if for $t_k = \|M_k\|_{op}$ (the operator norm) we have $\inf_{k \in \mathbb{N}} \frac{t_{k+1}}{t_k} \equiv \lambda > 1$. The sequence \mathcal{Z} is *uniformly discrete*, if there exists $\tau > 0$ such that every set Z_k is τ -separated. Under the assumption that \mathcal{M} is lacunary and \mathcal{Z} is uniformly discrete, [25] showed that if X is the support of a δ -decaying measure in the sense of (2.23), then $E(\mathcal{M}, \mathcal{Z}) \cap X$ is a winning set in X for Schmidt's game.

Using similar arguments for the proof, we to consider the following weaker condition in our weaker setting: In fact, assume there exists $0 < \bar{\delta} < \delta$ such that for all $r \in \mathbb{R}$ and $b > 0$ we have

$$|\{k \in \mathbb{N} : \log(t_k/\tau_k) \in (r - b, r]\}| \leq e^{\bar{\delta}b}. \quad (3.2)$$

Theorem 3.5. *Let $X \subset \mathbb{R}^n$ be the support of a δ -decaying measure μ , \mathcal{M} and \mathcal{Z} be as above satisfying (3.2). Then, $F(E(\mathcal{M}, \mathcal{Z}) \cap X)$ is a ψ_s -weakly-winning set in $F(X)$, for any bi-Lipschitz map $F : \mathbb{R}^n \rightarrow F(\mathbb{R}^n)$.*

If μ satisfies moreover a power law with respect to the exponent $\tau \geq \delta$, then $E(\mathcal{M}, \mathcal{Z}) \cap X$ and hence $F(E(\mathcal{M}, \mathcal{Z}) \cap X)$ are of Hausdorff-dimension τ by Theorem 2.10.

Proof. For $Y \subset \mathbb{R}^n$ we let $M_k^{-1}(Y) \subset \mathbb{R}^n$ be the preimage under $M_k \in \mathcal{M}$. Let $v_k \in \mathbb{R}^n$ be the unit vector such that $\|M_k v_k\| = t_k$ and if $V_k \equiv \{M_k v_k\}^\perp$ is the subspace orthogonal to $M_k v_k$, let $W_k \equiv M_k^{-1}(V_k)$. Note that for all $x \in \mathbb{R}^n$ we have $\|x\| \geq \|M_k x\|/t_k$. Hence, for distinct points $y_1, y_2 \in Z_k$, we have

$$\|M_k^{-1}(B(y_1, \tau_k/4)) - M_k^{-1}(B(y_2, \tau_k/4))\| \geq \frac{\tau_k - 2\tau_k/4}{t_k} = \frac{\tau_k}{2t_k}. \quad (3.3)$$

For $k \in \mathbb{N}$ and $y \in Z_k$ we therefore define the subsets $Y(y) \equiv (M_k^{-1}(y) + W_k) \cap M_k^{-1}(B(y, \tau_k/4))$. Define for $k \in \Lambda \equiv \mathbb{N}$ the resonant set

$$R_k \equiv \{x \in Y(y_l) : y_l \in Z_l \text{ and } \log(t_l) - \log(\tau_l) \leq \log(k)\}$$

with the size $s_k = \log(k) + \log(12)$. Then $\mathcal{F} = \{\mathbb{R}^n, X, \mathbb{N}, R_k, s_k, \leq_s, \psi_s\}$ is nested and discrete. We show that μ is $(\psi_s, f, \log(2))$ -decaying with respect to \mathcal{F} , where $f(b, s) = ce^{\bar{\delta}b}e^{-\delta s}$ for some constant $c > 0$.

In fact, given a closed ball $B = B(x, e^{-r}) \subset \mathbb{R}^n$ with $x \in X$, for every $k \in \mathbb{N}$ with $s_k \leq r$, it follows from (3.3) that at most one of the sets $Y(y)$, $y \in Z_k$, can intersect $B(x, 2e^{-r})$. Moreover, for $b > 0$, the number of $k \in \mathbb{N}$ with $\log(t_k) - \log(\tau_k) + \log(12) \in (r - b, r]$ is bounded by $e^{\bar{\delta}b}$ by (3.2). Thus, for $b > b_* \geq \log(2)$, there exist at most $N = \lfloor e^{\bar{\delta}b} \rfloor$ affine hyperplanes L_1, \dots, L_N such that

$$B \cap \mathcal{N}_{e^{-(r+s)}}(R(r, b)) \subset B \cap \bigcup_{i=1}^N \mathcal{N}_{e^{-(r+s)}}(L_i).$$

Since μ is δ -decaying, we have

$$\mu(B \cap \mathcal{N}_{e^{-(r+s)}}(R(r, b))) \leq \sum_{i=1}^N \mu(B \cap \mathcal{N}_{e^{-(r+s)}}(L_i)) \leq e^{\bar{\delta}b} \cdot ce^{-\delta s} \mu(B) = f(b, s) \mu(B).$$

Since $\bar{\delta} < \delta$, for every $n_* \in \mathbb{N}$, $d_* \geq \log(2)$ there exists a $\bar{b}_* = \bar{b}_*(d_*, n_*) \geq 2d_*$ such that $f(n_*b + d_*, b - 2d_*) \leq c < 1$ for all $b > \bar{b}_*$. By Proposition 2.9 and since $\bar{\psi}_s$ is $\log(2)$ -contracting with respect to \mathcal{F} , X is (\bar{b}_*, d_*, n_*) -diffuse with respect to \mathcal{F} . Hence, by Theorem 2.4 and Proposition 2.5, $F(\text{Bad}(\mathcal{F}))$ is ψ_s -weakly-winning for any L -bi-Lipschitz map with $2L \leq d_*$.

Finally, let $x \in \mathbf{Bad}(\mathcal{F})$, that is, there exists $c < \infty$ such that $d(x, Y(y)) \geq e^{-(s_k+c)} \geq \tau_k e^{-c-\log(12)}/t_k \equiv \bar{c}\tau_k/t_k$ for every $k \in \mathbb{N}$ and $y \in Z_k$. Assume that $M_k x \in B(y, \tau_k/4)$. Then, $x \in \mathcal{N}_{\bar{c}\tau_k/t_k}(M_k^{-1}(y) + W_k)^C \cap M_k^{-1}(B(y, \tau_k/4))$ and we can write the vector $v = x - M_k^{-1}(y)$ as $v = w + \tilde{c}\tau_k/t_k v_k$ with $w \in W_k$ and $\tilde{c} \geq \bar{c}$. Hence, since $M_k W_k$ is orthogonal to $M_k v_k$,

$$\|M_k x - y_k\| = \|M_k v\| = \|M_k w + \bar{c}/t_k M_k v_k\| \geq \bar{c}\tau_k/t_k \|M_k v_k\| = \bar{c}\tau_k,$$

so that $M_k x \notin B(y_k, \bar{c}\tau_k)$. This shows that $\mathbf{Bad}(\mathcal{F}) \subset E(\mathcal{M}, \mathcal{Z}) \cap X$ which implies that $E(\mathcal{M}, \mathcal{Z}) \cap X$ is ψ_s -weakly-winning. \square

3.4. The geodesic flow in \mathbb{H}^{n+1} . Let \mathbb{H}^{n+1} be the upper half-space model of the real hyperbolic $(n+1)$ -space. Although this example is covered by Subsection 3.6, we present a simplified proof of the following result due to [1, 4, 5, 19, 20, 28]. Instead of showing the existence of strategies for Schmidt's game, which are strongly adapted to the interplay between \mathbb{H}^{n+1} and its boundary $\bar{\mathbb{R}}^n \equiv \mathbb{R}^n \cup \{\infty\}$, we verify Condition (2.22).

Theorem 3.6. *Let Γ be a lattice in the isometry group $I(\mathbb{H}^{n+1})$ of \mathbb{H}^{n+1} with exactly one cusp at infinity. Then the endpoints in \mathbb{R}^n of lifts of bounded geodesic rays in \mathbb{H}^{n+1}/Γ is absolute winning (in the sense of McMullen); hence of Hausdorff-dimension n .*

For details and background of the proof we refer to Subsection 3.6.

Proof. Identify $\mathbb{R}^n \subset \mathbb{R}^n \times \{0\} \cup \{\infty\}$ as a subset of the visual boundary of \mathbb{H}^{n+1} and let $\Omega = \mathbb{R}^n \times (0, \infty)$. Let $\Gamma_0 = \text{Stab}_\Gamma(\infty)$ be the stabilizer of Γ at ∞ . It is well known that Γ determines a cusp neighborhood (see [32]), which in turn determines a countable family of disjoint Euclidean balls $E_{[\varphi]}$, $[\varphi] \in \Gamma/\Gamma_0$, tangent to \mathbb{R}^n in the point $x_{[\varphi]} = \varphi(\infty)$ and of radius $r_{[\varphi]}$. A ray in \mathbb{H}^{n+1} with endpoint in \mathbb{R}^n is bounded in \mathbb{H}^{n+1}/Γ if and only if it does not penetrate the disjoint Euclidean balls $E_{[\varphi]}$ too deep with respect to the radii $r_{[\varphi]}$. Hence, for $k \in \Lambda \equiv \mathbb{N}$ define the resonant set $R_k \equiv \{x_{[\varphi]} : [\varphi] \in \Gamma/\Gamma_0 \text{ such that } -\log(r_{[\varphi]}) \leq k\} \subset \mathbb{R}^n = X$ with size $s_k = k$. Let $\mathcal{F} = (\mathbb{N}, R_k, s_k)$, which is nested and discrete. Note that a ray as above is bounded in \mathbb{H}^{n+1}/Γ if and only if its endpoint in \mathbb{R}^n is not too close to the orbit points $x_{[\varphi]}$ with respect to the radii $r_{[\varphi]}$, that is, if and only if its endpoint belongs to $\mathbf{Bad}(\mathcal{F})$. Moreover, for every two distinct tangency points $x_{[\varphi]}$ and $x_{[\bar{\varphi}]}$ of radius $r_{[\bar{\varphi}]} \geq r_{[\varphi]} \geq e^{-k}$ we have that

$$d(x_{[\varphi]}, x_{[\bar{\varphi}]}) > r_{[\varphi]} \geq e^{-k} = e^{-s_k},$$

since the Euclidean balls $E_{[\varphi]}$ and $E_{[\bar{\varphi}]}$ are disjoint. Hence, (2.22) is satisfied and since \mathbb{R}^n is complete and β -diffuse ($\beta = \log(3)$), the proof now follows from Proposition 2.8. \square

Taking $\Gamma = SL(2, \mathbb{Z}) \subset I(\mathbb{H}^2)$ the modular group, observe that, with the standard cusp neighborhood $\mathbb{R} \times (1, \infty)$, the tangency points of the disjoint Euclidean balls with their radii as above give precisely the sets $B_{p/q} = B(p/q, 1/(2q^2))$ for $p/q \in \mathbb{Q}$ (with p, q coprime). Thus, one can easily show that $\mathbf{Bad}(\mathcal{F}) = \mathbf{Bad}(1)$ and we again have that the set of badly approximable real numbers in \mathbb{R} are of Hausdorff-dimension one.

3.5. The Bernoulli shift Σ^+ . For $n \geq 1$, let $\Sigma^+ = \{0, \dots, n\}^{\mathbb{N}}$ be the set of one-sided sequences in symbols from $\{0, \dots, n\}$. Let $T = \sigma$ denote the shift and let d^+ be the metric given by $d^+(w, \bar{w}) \equiv e^{-\min\{i \geq 1 : w(i) \neq \bar{w}(i)\}}$ for $w \neq \bar{w}$ and $d(w, w) \equiv 0$.

Fix a periodic word $\bar{w} \in \Sigma^+$ of period $p \in \mathbb{N}$. Consider the set

$$S_{\bar{w}} = \{w \in \Sigma^+ : \exists c = c(w) < \infty \text{ such that } T^k w \notin B(\bar{w}, 2^{-(p+c+1)}) \text{ for all } k \in \mathbb{N}\}.$$

Theorem 3.7. $S_{\bar{w}}$ is absolute winning (in the sense of McMullen) and of Hausdorff-dimension $\log(n)$.

Remark. In particular, the intersection $\bigcap S_{\bar{w}}$ over all periodic words $\bar{w} \in \Sigma^+$ is $(\psi_s, 1)$ -absolute winning and therefore of Hausdorff-dimension $\log(n)$. Note that the Morse-Thue sequence w in $\{0, 1\}^{\mathbb{N}}$ is a particular example of a word in $\bigcap_{\bar{w}} S_{\bar{w}}$. In fact, w does not contain any subword of the form WWa where a is the first letter of the subword W ; for details and more general words in $\bigcap_{\bar{w}} S_{\bar{w}}$, we refer to an earlier work of the author [29].

Proof. For $k \in \mathbb{N}$ and $w_k \in \{0, \dots, n\}^k$, let $\bar{w}_k \in \Sigma^+$ denote the word $\bar{w}_k = w_k \bar{w}$. Let $\Lambda \equiv \mathbb{N}_0$ and consider the resonant sets

$$R_0 = \{\bar{w}\}, \quad R_k = \{\bar{w}_l \in \Sigma^+ : w_l \in \{1, \dots, n\}^l, l \leq k\} \cup R_0, \text{ for } k \in \mathbb{N}$$

which we give the size $s_k = p + k + 1$. Then, $\mathcal{F} = (\mathbb{N}_0, R_k, s_k, \leq_s, \psi_s)$ is nested and discrete. Let $\bar{w}_m \in R_m$ and $\bar{w}_l \in R_l$ with $m \leq l$. Assume that

$$d^+(\bar{w}_m, \bar{w}_l) \leq \min\{e^{-s_m}, e^{-s_l}\} = e^{-(p+l+1)}.$$

Then, $\bar{w}_l(1) \dots \bar{w}_l(p+l) = \bar{w}_m(1) \dots \bar{w}_m(p+l)$ and by definition of \bar{w}_l and \bar{w}_m ,

$$\bar{w}_l(l+kp+i) = \bar{w}_l(l+i) = \bar{w}_m(l+i) = \bar{w}_m(l+kp+i),$$

for $0 \leq i < p$ and $k \in \mathbb{N}_0$, so that $\bar{w}_l = \bar{w}_m$. This shows that (2.22) is satisfied. Moreover, (Σ^+, d^+) is β -diffuse for $\beta = 1$. By Proposition 2.8 we know that $\mathbf{Bad}(\mathcal{F})$ is $(\psi_s, 1)$ -winning. Note moreover that the probability measure $\mu = \{1/n, \dots, 1/n\}^{\mathbb{N}}$ satisfies

$$n^{-(l+1)} \leq \mu(B(w, e^{-t})) \leq n^{-l}, \quad \text{for } l \leq t \leq l+1.$$

Hence, μ satisfies a power law with respect to the exponent $\log(n)$ and $\mathbf{Bad}(\mathcal{F})$ is of Hausdorff-dimension $\log(n)$ by Theorem 2.10.

Finally, we have $\mathbf{Bad}(\mathcal{F}) = S_{\bar{w}}$. In fact, $d^+(T^{k-1}w, \bar{w}) \leq e^{-(p+c+1)}$ for some $c \in \mathbb{N}$ if and only if $w(k) \dots w(k+p+c) = \bar{w}(1) \dots \bar{w}(p+c)$. Thus, for $w_k = w(1) \dots w(k)$ and $\bar{w}_k = w_k \bar{w}$ we have $d^+(w, \bar{w}_k) \leq e^{-(p+k+c+1)}$ if and only if $w \in B(\bar{w}_k, e^{-(s_k+c)}) \subset \psi_s(R_k, s_k + c)$. \square

3.6. The geodesic flow in CAT(-1)-spaces. We discuss the example of badly approximable limit points in CAT(-1)-spaces in more details. If GX denotes the space of geodesic rays in a proper geodesic CAT(-1) metric space X , then the semigroup \mathbb{R}^+ acts on GX via the geodesic flow (g^s) which itself acts by reparameterization,

$$g^s(\gamma)(t) = \gamma(t+s).$$

Given a family of suitable (convex) resonant sets $R_\lambda \subset X, \lambda \in \Lambda$, we can ask about the rays which avoid contractions or have bounded penetrations in neighborhoods of the resonant sets. The behavior of penetration lengths of geodesic rays in convex subsets of X leads to a model of Diophantine approximation in CAT(-1)-spaces, developed by Hersonsky, Paulin and Parkkonen in [11, 12, 13, 23] and slightly different considered by Mayeda, Merrill [19]. With respect to the visual metric d_o (where o is a base point), we thereby translate our problem to the compact metric space $(\partial_\infty X, d_o)$ and, since d_o is a metric on the set of asymptotic rays, we induce suitable resonant sets \bar{R}_λ in $\partial_\infty X$ related to the resonant sets R_λ .

We begin by recalling the preliminaries which will be needed and introduce the model of Diophantine approximation.

3.6.1. Preliminaries. For a general reference and further details we refer to [3]. In the following, (X, d) denotes a proper geodesic $\text{CAT}(-1)$ metric space and $\partial_\infty X$ its visual boundary, that is, the set of equivalence classes of asymptotic rays. Equip $X \cup \partial_\infty X$ with the cone topology. Given two points $x, y \in X \cup \partial_\infty X$ we denote by $[x, y]$ the unique geodesic segment from x to y . For three points $o, x, y \in X \cup \partial_\infty X$, let

$$(x, y)_o \equiv \frac{1}{2}(d(o, x) + d(o, y) - d(x, y))$$

be the Gromov-product at o and for $\xi, \eta \in \partial_\infty X$, let $(\xi, \eta)_o \equiv \lim_{t \rightarrow \infty} (\gamma_{o, \xi}(t), \gamma_{o, \eta}(t))_o$ be the extended Gromov-product at o , where $\gamma_{o, \xi} \equiv [o, \xi]$. For $o \in X$, we define $d_o : \partial_\infty X \times \partial_\infty X \rightarrow [0, \infty)$ by $d_o(\xi, \xi) \equiv 0$ and for $\xi \neq \eta$ by

$$d_o(\xi, \eta) \equiv e^{-(\xi, \eta)_o},$$

called the *visual metric* at o . Then $(\partial_\infty X, d_o)$ is a compact metric space. For $\xi \in \partial_\infty X$ and $y \in X$, the *Busemann function* $\beta = \beta_{\xi, y} : X \rightarrow \mathbb{R}$ (with respect to y) is defined by

$$\beta(x) \equiv \lim_{t \rightarrow \infty} d(x, \gamma_{y, \xi}(t)) - t,$$

which is continuous and convex on X and $\beta(y) = 0$. The level sets of $\beta_{\xi, y}$ are called *horospheres* at ξ and the sublevel sets are called *horoballs* at ξ (with respect to y).

Let $\Gamma \subset I(X)$ be a discrete subgroup of the isometry group $I(X)$ of X . The *limit set* $\Lambda\Gamma$ of Γ is the compact subset $\overline{\Gamma \cdot x} \cap \partial_\infty X$ of $\partial_\infty X$, for any $x \in X$. If $\Lambda\Gamma$ contains at least two points, then $\mathcal{C}\Gamma$ denotes the convex hull of $\Lambda\Gamma$. Note that every isometry $\varphi \in I(X)$ extends to a homeomorphism on $\partial_\infty X$.

Every $\text{CAT}(-1)$ space is a (tripod) δ -hyperbolic space for some $\delta > 0$, that is, for all $o \in X$ and $x, y \in X$ or $x, y \in \partial_\infty X$, we have

$$p \in [o, x], q \in [o, y] \text{ with } d(o, p) = d(o, q) \leq (x, y)_o \implies d(p, q) \leq \delta. \quad (3.4)$$

Moreover, there exists a $\kappa > 0$, depending only on δ , such that for all $o \in X$ and $\xi, \eta \in \partial_\infty X$,

$$0 \leq d(o, [\xi, \eta]) - (\xi, \eta)_o \leq \kappa. \quad (3.5)$$

We will make use of the following results.

Lemma 3.8 ([22], Lemma 2.1). *Let $x, y \in X$ and for $z \in X \cup \partial_\infty X$ let $\gamma = [x, z]$. Then, for all $t \in [0, d(x, z)]$,*

$$d(\gamma(t), [y, z]) \leq \frac{1}{2}e^{d(x, y) - t}.$$

If $\varepsilon > 0$ and α is a geodesic segment, let $\mathcal{N}_\varepsilon(\alpha)$ be the closed ε -neighborhood of α which is itself convex. As a consequence of Lemma 3.8, we prove that a ray which penetrates in the D -neighborhood of a geodesic segment for a sufficiently long time must also penetrate in its ε -neighborhood.

Lemma 3.9. *Fix $D \geq \varepsilon > 0$. Let γ and α be two geodesics in X such that $d(\gamma(-L), \alpha) \leq D$ and $d(\gamma(L), \alpha) \leq D$, where $L \geq 2(D - \log(\varepsilon))$. Then there exists a constant $c = c(D, \varepsilon) \leq D - \log(\varepsilon)$ such that $\gamma([-L + c, L - c]) \subset \mathcal{N}_\varepsilon(\alpha)$.*

Proof. First, consider the case when γ and α do not intersect. Let p and $q \in \alpha$ be the closest points of $a = \gamma(L)$ and $b = \gamma(-L)$ respectively at distance at most D on α . We subdivide the quadrilateral (a, b, p, q) into two geodesic triangles (a, b, p) and (b, p, q) with a connecting geodesic $\tilde{\gamma} = [b, p]$. Note that $\tilde{L} \equiv d(b, p) \geq 2L - D$. For $t \in [0, \tilde{L}]$, let $b_t \in \gamma$

and $q_t \in \alpha$ be the closest points of $\tilde{\gamma}(t)$ on γ and α respectively. Let $t_0 = D - \log(\varepsilon)$. From Lemma 3.8 we have $d(\tilde{\gamma}(t_0), q_{t_0}) \leq e^{-t_0} e^D / 2 = \varepsilon / 2$, as well as,

$$d(\tilde{\gamma}(t_0), b_{t_0}) \leq \frac{1}{2} e^{-(\tilde{L}-t_0)} e^D \leq \frac{1}{2} e^{-2L+D+\log(\varepsilon)} \leq \frac{\varepsilon}{2},$$

since $L \geq 2(D - \log(\varepsilon))$. Thus, $d(b_{t_0}, \alpha) \leq \varepsilon$. Note that $d(\gamma(L), b_{t_0}) \leq t_0$ by properties of the closest point map. In the same way, we define a_{t_0} for the two geodesic triangles (a, b, q) and (a, p, q) . Similarly, we obtain that also $d(a_{t_0}, \alpha) \leq \varepsilon$ with $d(\gamma(-L), a_{t_0}) \leq t_0$. Therefore, we see by convexity of the distance function that $\gamma([-L + t_0, L - t_0]) \subset [a_{t_0}, b_{t_0}] \subset \mathcal{N}_\varepsilon(\alpha)$.

The case when γ and α intersect follows from the same arguments (and is simpler). \square

Lemma 3.10 ([22], Lemma 2.9). *Let C_0 be a horoball in X and $o \in X - C_0$. Then, for two geodesic rays starting in o and entering in C_0 at x and \bar{x} respectively, we have*

$$d(x, \bar{x}) \leq 2 \log(1 + \sqrt{2}) \equiv c_0.$$

If $\tau \geq 0$ and $C_0 = \beta^{-1}((-\infty, 0])$ is a (closed) horoball with respect to the Busemann function β , let $C_0[\tau] \equiv \beta^{-1}((-\infty, -\tau]) = \{x \in C_0 : d(x, \partial C_0) \geq \tau\} \subset C_0$ denote the horoball shrunk by the factor τ . Let $o \in X - C_0$ and assume that for $\xi \in \partial_\infty X$ the ray $\gamma_{o,\xi}$ enters in C_0 . Define the *shrinking parameter* of ξ by $s(\xi) = \sup\{\tau \in [0, \infty] : \gamma_{o,\xi} \cap C_0[\tau] \neq \emptyset\}$. Then the ray $\gamma_{o,\xi}$ penetrates the horoball C_0 for a long time if and only if it enters deeply into C_0 , that is, its shrinking parameter is large.

Lemma 3.11. *Let $o \in X - C_0$. Assume that for $\xi \in \partial_\infty X$ the ray $\gamma_{o,\xi}$ enters in C_0 at time $t \geq 0$ and leaves at time $t + p$, $0 < p < \infty$. Let $s \geq 0$ be the shrinking parameter of ξ . Then*

$$2s - c_0 \leq p \leq 2s + 2c_0.$$

Proof. Let C_0 be based at the point $\eta \in \partial_\infty X$, $\eta \neq \xi$, and let $d_o = d(o, C_0) \geq 0$ such that $\gamma_{o,\eta}(d_o) \in \partial C_0$. Note that the function $s \mapsto \beta \circ \gamma_{o,\xi}(s)$ is continuous and convex. Hence, there exists a point $\xi_s \equiv \gamma_{o,\xi}(t + p_1)$ on $\partial C_0[s] = \beta^{-1}(-s)$. By Lemma 3.10, we have $d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d_o)) \leq c_0$, as well as $d(\xi_s, \gamma_{o,\eta}(d_o + s)) \leq c_0$. Note that for all $\tau \geq 0$, $\gamma_{o,\eta}(d_o + \tau)$ is the closest point of o to $\partial C_0[\tau]$. Hence, $d_o \leq t \leq d_o + c_0$ as well as

$$d_o + s \leq t + p_1 \leq d_o + c_0 + s.$$

Starting with the point $\tilde{o} = \gamma_{o,\xi}(t + p) \in \partial C_0$ with $d_{\tilde{o}} = d(\tilde{o}, C_0) = 0$, we obtain in the same way that $s \leq p_2 \equiv p - p_1 \leq s + c_0$, by Lemma 3.10. Thus,

$$\begin{aligned} 2s - c_0 \leq 2s + d_o - t &\leq p_1 + p_2 = p \\ &\leq d_o - t + 2s + 2c_0 \leq 2s + 2c_0, \end{aligned}$$

which finishes the proof. \square

3.6.2. A model of Diophantine approximation in negatively curved spaces. Let Γ be a discrete subgroup of the isometry group $I(X)$ of X . Recall that a subgroup $\Gamma_0 \subset \Gamma$ is called *convex cocompact* if $\Lambda\Gamma_0$ contains at least two points and the action of Γ_0 on the convex hull $CH\Gamma_0$ has compact quotient. We call Γ_0 *bounded parabolic* if Γ_0 is the stabilizer of a parabolic fixed point $\xi \in \Lambda\Gamma$, and if there exists a horoball C_0 at ξ such that the action of Γ_0 on ∂C_0 has compact quotient (hence, $\Lambda\Gamma_0 = \{\xi\}$). Moreover, we call Γ_0 *almost malnormal* if for all $\varphi \in \Gamma - \Gamma_0$ we have $\varphi.\Lambda\Gamma_0 \cap \Lambda\Gamma_0 = \emptyset$.

Let $\Gamma_i \subset \Gamma$, $i = 1, 2$, be an almost malnormal subgroup in Γ of infinite index. We treat the following two cases simultaneously:

1. Let Γ_1 be bounded parabolic and C_1 be the horoball as in the definition.
2. Let Γ_2 be convex-cocompact and $C_2 = \mathcal{C}\Gamma_2$ be the convex hull of Γ_2 , where we assume that either
 - a) C_2 is a geodesic line, or,
 - b) every image $\varphi.\Lambda\Gamma_2$, $[\varphi] \in \Gamma/\Gamma_2$, is contained in a metric sphere (with respect to d_o , where o is a base point as below).

Choose a base point $o \in X - \Gamma.C_i$ and define the sets

$$S_1 = \{\xi \in \Lambda\Gamma : \exists c = c(\xi) < \infty \text{ such that the length}^{10} L(\gamma_{o,\xi}(\mathbb{R}^+) \cap \varphi C_1) \leq c \text{ for all } [\varphi] \in \Gamma/\Gamma_1\},$$

and similarly S_2 , replacing φC_1 by $\mathcal{N}_\varepsilon(\varphi C_2)$ for some fixed $\varepsilon > 0$.

Example. Let C_2 be a totally geodesic submanifold (of dimension at most n) in \mathbb{H}^{n+1} , the hyperbolic ball model, and $o = 0$ be the center of \mathbb{H}^{n+1} . Then C_2 is contained in a subspace isometric to \mathbb{H}^n and the boundary of this subspace is a metric sphere (with respect to d_o). Hence, $\partial_\infty C_2 = \Lambda\Gamma_2$ and every image $\varphi.\Lambda\Gamma_2$ is contained in metric spheres.

Note that, since Γ_i is almost malnormal, we have $\Gamma_i = \text{Stab}_\Gamma(C_i)$. Moreover, C_i is (ε, T) -embedded, that is, for every $\varepsilon > 0$ there exists $T = T(\varepsilon) \geq 0$ such that for all $\varphi \in \Gamma - \Gamma_i$ we have that $\text{diam}(\mathcal{N}_\varepsilon(\mathcal{C}\Gamma_i) \cap \varphi(\mathcal{N}_\varepsilon(\mathcal{C}\Gamma_i))) \leq T$; see [23]. In the first case, we therefore assume, after shrinking C_1 , that the images ψC_1 , $[\psi] \in \Gamma/\Gamma_1$, are disjoint.

Furthermore, when $\Gamma \subset I(\mathbb{H}^n)$ is non-elementary, we consider a separated third case:

3. Let $x \in X = \mathbb{H}^n$ such that $x \notin \Gamma.o$, let Γ_3 be the stabilizer of x in Γ and take $C_3 = \{x\}$ in the following. For sufficiently large $t_0 > 0$ we set

$$S_3 = \{\xi \in \Lambda\Gamma : \exists c = c(\xi) > 0 \text{ such that } d(\gamma_{o,\xi}(t), \varphi(x)) \geq c \text{ for all } [\varphi] \in \Gamma/\Gamma_3 \text{ and } t > t_0\}.$$

For the respective cases, $i = 1, 2, 3$, denote the quadruple of data by

$$\mathcal{D}_i = (X, \Gamma, C_i, o).$$

For $r = [\varphi] \in \Gamma/\Gamma_i$ we define

$$D_i(r) = d(o, \varphi C_i)$$

which does not depend on the choice of the representative φ of r .

Remark. For $r = [\varphi] \in \Gamma/\Gamma_i$, let $\mathcal{S}_o(\varphi C_i) \subset \partial_\infty X$ be the shadow of the set ψC_i with respect to the base point o . Using (3.5), Lemma 3.8 and 3.10, one can show that if $D_i(r)$ is sufficiently large, the size (diameter with respect to d_o) of the shadows $\mathcal{S}_o(\varphi C_i)$ is comparable to the quantity $e^{-D_i(r)}$. We therefore consider the *approximation function* $f_i(r) = e^{D_i(r)}$ as a renormalization of the size of the shadows.

Note that the set $\{D_i(r) : r \in \Gamma/\Gamma_i\}$ is discrete and unbounded:

Lemma 3.12. *For every $D \geq 0$ there are only finitely many elements $r \in \Gamma/\Gamma_i$ such that $D_i(r) \leq D$ and there exists an $r \in \Gamma/\Gamma_i$ such that $D_i(r) > D$.*

Proof. For the second case, the proof follows from Lemma 3.1 and 3.2 in [23] with the difference that we do not consider the stabilizer of o in Γ (which is only a finite group in our case). The arguments of the proof also work for the first case. The third case follows since Γ is discrete and non-elementary. \square

¹⁰ Note that since C_i is convex, $\gamma_{o,\xi}(\mathbb{R}^+) \cap \varphi C_i$ is the image of a connected geodesic segment.

Now, for $i = 1, 2, 3$ and for $\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i$ define the *approximation constant*

$$c_i(\xi) = \liminf_{r=[\varphi] \in \Gamma/\Gamma_i} e^{D_i(r)} d_o(\xi, \varphi.\Lambda\Gamma_i),$$

where we replace $\varphi.\Lambda\Gamma_i$ by $\varphi(x)_\infty \equiv \gamma_{o, \varphi(x)}(\infty)$ in the third case. If $c_i(\xi) = 0$ then ξ is called *well approximable*, otherwise it is called *badly approximable* (with respect to \mathcal{D}_i). Define the set of badly approximable limit points by

$$\mathbf{Bad}(\mathcal{D}_i) = \{\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i : c_i(\xi) > 0\} \subset \Lambda\Gamma.$$

For the following result, we consider $X = \bar{X} = \Lambda\Gamma$ in $\partial_\infty X$ with respect to the induced metric of d_o and let \mathcal{S} be the set of metric spheres in $\Lambda\Gamma$.

Theorem 3.13. *We have $S_i = \mathbf{Bad}(\mathcal{D}_i)$ for $i = 1, 2, 3$. Moreover, if the limit set $\Lambda\Gamma$ of Γ is β -diffuse for the Cases 1. and 2.a), then S_i is absolute winning (in the sense of McMullen). If $\Lambda\Gamma$ is b_* -diffuse with respect to \mathcal{S} (in the sense of (2.13)), then S_2 is ψ_s -winning for the Case 2.b). If $\Lambda\Gamma$ is the support of a locally finite Borel measure satisfying a power law with respect to the exponent τ , then for any bi-Lipschitz map $F : \Lambda\Gamma \rightarrow F(\Lambda\Gamma)$, $F(S_3)$ is ψ_s -weakly-winning in $F(\Lambda\Gamma)$ and of Hausdorff-dimension τ .*

The first case has been considered by [19] in the case of δ -hyperbolic spaces. However, they used a different definition of badly approximable limit points using the size of the shadows of the disjoint horoballs. A result related to the third case is due to [10].

Remark. Recall that if $X = \mathbb{H}^{n+1}$ and Γ is a non-elementary finitely generated Kleinian group, then $\Lambda\Gamma \subset S^n$ is uniformly perfect; see [15]. In particular, $\Lambda\Gamma$ is β -diffuse for some $\beta > 0$ by Lemma 2.7.

Remark. The proof of Case 1 and 2 essentially uses that C_i is (ε, T) -embedded for $i = 1, 2$. Moreover, Case 3 holds true for X a Riemannian manifold of pinched negative curvature.

Remark. Note that the visual distance at a point $o \in X$ is comparable to the Hamenstädt metric with respect to a horoball H_0 : For every compact subset K of $\partial_\infty X - \partial_\infty H_0$, there exists a constant $c_K > 0$ such that for all $\xi, \eta \in K$,

$$c_K^{-1} d_o(\xi, \eta) \leq d_{H_0}(\xi, \eta) \leq c_K d_o(\xi, \eta);$$

see [11], Lemma 2.3. We therefore focus only on the visual distance in our settings, which can however, up to further requirements (see [23]), be replaced by the Hamenstädt metric.

3.6.3. A measure on $\Lambda\Gamma$. Let $X = \mathbb{H}^{n+1}$ be the hyperbolic ball model and let $o = 0$ be the center. Note that the visual distance d_o is equivalent to the angle metric on the unit sphere $S^n = \partial_\infty \mathbb{H}^{n+1}$. Hence, if Γ is of the *first kind*, that is $\Lambda\Gamma = \partial_\infty \mathbb{H}^{n+1}$, then $\Lambda\Gamma = S^n$ is b_* -diffuse for $b_* > \log(3)$ and the Lebesgue measure on S^n satisfies a power law with respect to the visual metric d_0 and the exponent n . More generally, recall that the *critical exponent* of a discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ is given by

$$\delta(\Gamma) \equiv \inf \left\{ s > 0 : \sum_{\varphi \in \Gamma} e^{-sd(x, \varphi(x))} < \infty \right\},$$

for any $x \in \mathbb{H}^{n+1}$. Associated to Γ , there is a canonical measure, the *Patterson-Sullivan* measure μ_Γ , which is a $\delta(\Gamma)$ -conformal probability measure supported on $\Lambda\Gamma$. For a precise definition we refer to [21]. If Γ is non-elementary and convex-cocompact, then $\delta(\Gamma)$ equals the Hausdorff-dimension of $\Lambda\Gamma$; in particular, the Patterson-Sullivan measure $\mu_{\Gamma, o}$ (at o) satisfies a power law with respect to the exponent $\delta(\Gamma)$. There are various further results concerning the Patterson-Sullivan measure. Here, we point out the following.

Regarding Case 1, it is shown in [19] that if Γ is a non-elementary geometrically finite Kleinian group, the set of limit points, which correspond to geodesics starting in o and projecting to bounded geodesics in \mathbb{H}^{n+1}/Γ , has dimension $\delta(\Gamma)$. In particular, S_1 contains this set and is thus of dimension $\delta(\Gamma)$.

For the second case, let $\mathcal{H}(\Gamma) \equiv \{S \cap \Lambda\Gamma : S \text{ is a sphere in } S^n \text{ of codimension at least } 1\}$ which contains the set \mathcal{S} . A finite Borel measure ν on S^n is called $\mathcal{H}(\Gamma)$ -friendly, if ν is Federer and there exist $\delta, r_0, c > 0$ such that for all $S \cap \Lambda\Gamma \in \mathcal{H}(\Gamma)$ and $s > 0$ we have $\nu(B(\xi, r) \cap \mathcal{N}_{sr}(S \cap \Lambda\Gamma)) \leq cs^\delta \nu(B(\xi, r))^{11}$ for all $0 < r < r_0$ and $\xi \in \Lambda\Gamma$.

Theorem 3.14 ([30], Theorem 2). *For every non-elementary convex cocompact discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ (without elliptic elements), such that $\Lambda\Gamma$ is not contained in a finite union of elements of $\mathcal{H}(\Gamma)$, the Patterson-Sullivan measure μ_Γ is $\mathcal{H}(\Gamma)$ -friendly.*

Now, if Γ is as in Theorem 3.14, then it follows as in Proposition 2.9 that $\Lambda\Gamma$ is b_* -diffuse with respect to \mathcal{S} for some $b_* > 0$ sufficiently large. Hence, for Case 2, $S_2 = \mathbf{Bad}(\mathcal{D}_2)$ is ψ_s -winning by Theorem 3.13. Moreover, since μ satisfies a power law, we see that S_2 is of Hausdorff-dimension $d_\mu(\Lambda\Gamma) = \delta(\Gamma) = \dim(\Lambda\Gamma)$ by Theorem 2.10.

Summarizing, we have the following.

Corollary 3.15. *If Γ is non-elementary geometrically finite Kleinian group in Case 1 or as in Theorem 3.14 in Case 2, then S_i is of Hausdorff-dimension $\delta(\Gamma)$ for $i = 1, 2$.*

3.6.4. The induced resonant sets. Note first that it suffices to consider rays starting at the base point o as we are interested in an asymptotic behavior. From the dynamical viewpoint, in the first two cases, a limit point $\xi \in \Lambda - \Gamma.\Lambda\Gamma_i$ is well or badly approximable depending on the behavior of the geodesic ray $\gamma_{o,\xi} = [o, \xi]$ with respect to its penetrations in $\Gamma.C_1$ or in the neighborhoods $\Gamma.\mathcal{N}_\varepsilon(C_2)$ (for some $\varepsilon > 0$) respectively.

Lemma 3.16. *Let $\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i$. For $i = 1, 2, 3$, there exists a sequence $r_n = [\varphi_n] \in \Gamma/\Gamma_i$, a sequence of times $t_n \rightarrow \infty$ and a sequence of lengths $p_n \rightarrow \infty$ such that*

1. $\gamma_{o,\xi}([t_n, t_n + p_n]) \subset \varphi_n(C_1)$,
2. $\gamma_{o,\xi}([t_n, t_n + p_n]) \subset \varphi_n(\mathcal{N}_\varepsilon(C_2))$,
3. $d(\gamma_{o,\xi}(t_n), \varphi_n(x)) \leq e^{-p_n} e^{-d(o, \varphi_n(x))}$,

for all sufficiently large $n \in \mathbb{N}$ if and only if

1. $d_o(\xi, \varphi_n.\Lambda\Gamma_1) \leq \bar{c}e^{-p_n/2} \cdot e^{-D_1(r_n)}$,
2. $d_o(\xi, \varphi_n.\Lambda\Gamma_2) \leq \bar{c}e^{-p_n} \cdot e^{-D_2(r_n)}$,
3. $d_o(\xi, \varphi_n(x)_\infty) \leq \bar{c}e^{-p_n} \cdot e^{-D_3(r_n)}$,

where $\bar{c} > 0$ is a universal constant; that is, if and only if $c_i(\xi) = 0$.

Proof. For the first case, let $r_n \in \Gamma/\Gamma_1$ such that $\gamma_{o,\xi}([t_n, t_n + p_n]) \subset \varphi_n(C_1)$. We assume that t_n is the entering and $t_n + p_n$ the exiting time. Then from Lemma 3.11, $p_n \leq 2s_n + 2c_0$, where s_n denotes the shrinking parameter of ξ in $\varphi_n C_1$. Moreover, note that if x_n is the closest point of o on $[\xi, \varphi_n.\Lambda\Gamma_1]$, we have

$$d(o, x_n) \geq d(o, \varphi_n(C_1[s_n])) = D_1(r_n) + s_n \geq D_1(r_n) + p_n/2 - c_0.$$

From (3.5) it follows that

$$d_o(\xi, \varphi_n.\Lambda\Gamma_1) = e^{-(\xi, \varphi_n.\Lambda\Gamma_1)_o} \leq e^{-d(o, x_n) + \kappa} \leq e^{-p_n/2 + (\kappa + c_0)} e^{-D_1(r_n)}.$$

¹¹ Note that there exist constants $c_1, c_2 > 0$ such that for all $\xi \in \Lambda\Gamma$ and $r > 0$ sufficiently small we have $B_{d_0}(\xi, c_1 r) \subset B(\xi, r) \subset B_{d_0}(\xi, c_2 r)$.

Conversely, let $d_o(\xi, \varphi_n \cdot \Lambda \Gamma_1) \leq c(p_n)e^{-D_1(r_n)}$ with $c(p_n) \leq \bar{c}e^{-p_n/2}$ where $\bar{c} > 0$ is sufficiently large. Set $t_n := D_1(r_n) + \delta$ and

$$t_n + \bar{p}_n := -\log(d_o(\xi, \varphi_n \cdot \Lambda \Gamma_1)) = (\xi, \varphi_n \cdot \Lambda \Gamma_1)_o \geq \log(\bar{c}) + D_1(r_n) + p_n/2.$$

Note that $\bar{p}_n \rightarrow \infty$ for $p_n \rightarrow \infty$ and let n be sufficiently large such that $t_n + \bar{p}_n > t_n$. Since $t_n + \bar{p}_n = (\xi, \varphi_n \cdot \Lambda \Gamma_1)_o$, we have from (3.4) that $d(\gamma_{o,\xi}(t_n + \bar{p}_n), \gamma_{o,\varphi_n \cdot \Lambda \Gamma_1}(t_n + \bar{p}_n)) \leq \delta$, and by convexity also, $d(\gamma_{o,\xi}(t_n), \gamma_{o,\varphi_n \cdot \Lambda \Gamma_1}(t_n)) \leq \delta$. Thus, we obtain that

$$\gamma_{o,\xi}([t_n, t_n + \bar{p}_n]) \subset \mathcal{N}_\delta(\gamma_{o,\varphi_n \cdot \Lambda \Gamma_1}([D_1(r_n) + \delta, \infty))) \subset \varphi_n(C_1).$$

The second case follows with similar arguments using Lemma 3.9. The proof can be found in [23], Lemma 4.1.

For the third case, from Lemma 3.1 in [10], there exist positive (universal) constants c_1, c_2, c_3 , such that for every $\psi(x), \varphi \in \Gamma$, with $d(o, \varphi(x)) \geq c_2$ (which we may assume if t_0 is sufficiently large) and for all $0 < R \leq c_3$ and $R \leq d(o, \varphi(x))$, we have

$$B_{d_o}(\varphi(x)_\infty, Re^{-d(o, \varphi(x))}) \subset \mathcal{S}_o(B(\varphi(x), R)) \subset B_{d_o}(\varphi(x)_\infty, c_1 Re^{-d(o, \varphi(x))}).$$

Here, $\mathcal{S}_o(B(\varphi(x), R))$ denotes the shadow at infinity of the metric ball $B(\varphi(x), R)$, which is disjoint to $\{o\}$. \square

For the first two cases we therefore consider a ray $\gamma_{o,\xi}$ to be badly approximable with respect to our resonant sets $R_{[\varphi]} = \varphi(C_i)$ or $R_{[\varphi]} = \varphi \mathcal{N}_\varepsilon(C_2)$ respectively, $[\varphi] \in \Gamma/\Gamma_i$, if and only if the sequence of penetrations lengths of $\gamma_{o,\xi}$ in the resonant sets is bounded by some length $L = L(\xi)$. While in the case of C_1 being a horoball, this is equivalent to saying that $\gamma_{o,\xi}$ avoids the shrunk horoballs $\varphi(C_1[s])$ for some $s = s(\xi)$, there is no appropriate analogue in the case of C_2 being a geodesic line. However, Lemma 3.16 allows us to induce suitable resonant sets on the limit set $\Lambda \Gamma$ as follows:

For $m \in \Lambda_i \equiv \mathbb{N}$ let (with $\varphi \cdot \Lambda \Gamma_i$ replaced by $\varphi(x)_\infty$, $\Lambda_3 \equiv \mathbb{N}_{[t_0]}$ in the third case)

$$\bar{R}_m^i \equiv \{\xi \in \varphi \cdot \Lambda \Gamma_i : [\varphi] \in \Gamma/\Gamma_i \text{ such that } D_i([\varphi]) \leq m\},$$

and define its size by $s_m^i = m + c_i$, where

$$c_1 \equiv \delta, \quad c_2 \equiv T + 2c(2\delta, \varepsilon/2) + \varepsilon, \quad c_3 \equiv 0,$$

and $c(2\delta, \varepsilon/2) \leq 2\delta - \log(\varepsilon/2)$ is the constant from Lemma 3.9 (and i stands for the respective case). Note that $(\Lambda \Gamma, d_o)$ is a complete metric space and define the nested and discrete family $\mathcal{F}_i = (\Lambda \Gamma, \Lambda \Gamma, \Lambda_i, R_m^i, s_m^i, \leq_s, \psi_s)$. It follows from Lemma 3.16 that

$$\mathbf{Bad}(\mathcal{D}_i) = \mathbf{Bad}(\mathcal{F}_i).$$

3.6.5. Verifying (b_*^s) for the Cases 1. and 2. We first show that condition (2.22) is satisfied for the Cases 1. and 2.a). For $[\bar{\varphi}], [\varphi] \in \Gamma/\Gamma_i$ let $\eta \in \varphi \cdot \Lambda \Gamma_i \subset \bar{R}_\lambda^i$ and $\bar{\eta} \in \bar{\varphi} \cdot \Lambda \Gamma_i \subset \bar{R}_{\bar{\lambda}}^i$. Assume that

$$d_o(\eta, \bar{\eta}) = e^{-(\eta, \bar{\eta})_o} \leq e^{-\max\{s_{\lambda(\eta)}, s_{\lambda(\bar{\eta})}\}} \leq e^{-\max\{D_i([\varphi]), D_i([\bar{\varphi}])\} - c_i}, \quad (3.6)$$

and set $D \equiv \max\{D_i([\varphi]), D_i([\bar{\varphi}])\}$. Equivalently, we have $(\eta, \bar{\eta})_o \geq D + c_i$ and by (3.4), we see that

$$d(\gamma_{o,\eta}(D + c_i), \gamma_{o,\bar{\eta}}(D + c_i)) \leq \delta.$$

Case 1: By definition, $c_1 = \delta$ and hence,

$$d(\gamma_{o,\varphi\eta}(D + \delta), \gamma_{o,\bar{\varphi}\eta}(D + \delta)) \leq \delta.$$

On the other hand, both the points $\gamma_{\psi\eta}(D+\delta)$ and $\gamma_{\bar{\psi}\eta}(D+\delta)$ are contained in the horoballs ψC_1 and $\bar{\psi} C_1$, at distance at least δ to the boundaries of the respective horoballs. Therefore, if φC_1 and $\bar{\varphi} C_1$ are disjoint, then

$$d(\gamma_{o,\eta}(D+\delta), \gamma_{o,\bar{\eta}}(D+\delta)) \geq 2\delta,$$

which is a contradiction. Hence, $[\varphi] = [\bar{\varphi}]$ and $\{\eta\} = \{\bar{\eta}\} = \psi.\Lambda\Gamma_1$.

Case 2.a): By definition, $c_2 = T + 2c(2\delta, \varepsilon/2) + \varepsilon$, and hence,

$$d(\gamma_{o,\eta}(D + T + 2c(2\delta, \varepsilon/2) + \varepsilon), \gamma_{o,\bar{\eta}}(D + T + 2c(2\delta, \varepsilon/2) + \varepsilon)) \leq \delta.$$

Since $D \geq (\psi C_2(\infty), \psi C_2(-\infty))_o$ we again have by (3.4) that

$$\gamma_{o,\eta}([D, D + T + 2c(2\delta, \varepsilon/2) + \varepsilon]) \subset \mathcal{N}_{2\delta}(\varphi C_2),$$

and the same is true for $\gamma_{o,\bar{\eta}}$. Therefore, by convexity of the distance function, we have

$$\gamma_{o,\eta}([D, D + T + 2c(2\delta, \varepsilon/2) + \varepsilon]) \subset \mathcal{N}_{2\delta}(\bar{\varphi} C_2),$$

and it follows from Lemma 3.9 that

$$\gamma_{o,\eta}([D, D + T + \varepsilon]) \subset \mathcal{N}_{\varepsilon/2}(\varphi C_2) \cap \mathcal{N}_{\varepsilon/2}(\bar{\varphi} C_2).$$

In particular, $\text{diam}(\mathcal{N}_{\varepsilon}(\varphi C_2) \cap \mathcal{N}_{\varepsilon}(\bar{\varphi} C_2)) \geq T + \varepsilon > T$ which is not possible since C_2 is (ε, T) -immersed. Hence, $[\varphi] = [\bar{\varphi}]$ and $\eta, \bar{\eta} \in \varphi.\Lambda\Gamma_2 = \{\varphi(C_2(\infty)), \varphi(C_2(-\infty))\}$. If $\eta \neq \bar{\eta}$, then $d_o(\eta, \bar{\eta}) = e^{-(\eta, \bar{\eta})_o} \geq e^{-D_2([\varphi])}$; a contradiction to (3.6).

Thus, if $\Lambda\Gamma$ is β -diffuse for some $\beta \geq 0$, we see by Proposition 2.8 that **Bad**(\mathcal{D}_i) is absolute winning (in the sense of McMullen).

Case 2.b) Note that by adding the constant $\log(4)$ to every size s_λ , we see by the above proof for Case 2.a) that the distance between two different images $\varphi C_2 \subset \bar{R}_\lambda^2$ and $\bar{\varphi} C_2 \subset \bar{R}_\beta^2$, $[\varphi], [\bar{\varphi}] \in \Gamma/\Gamma_2$, is bounded below by $d_o(\varphi C_2, \bar{\varphi} C_2) > 4 \cdot e^{-\max\{s_\lambda, s_\beta\}}$. Hence, given a metric ball $B = B_{d_o}(\xi, e^{-r})$, at most one image φC_2 , $[\varphi] \in \Gamma/\Gamma_2$ with $s_\lambda \leq r$, can intersect $B_{d_o}(\xi, 2e^{-r})$. For $b_* > 0$, we see that $B \cap \mathcal{N}_{e^{-(r+b_*)}}(R(r))$ is either empty or equals $B \cap \mathcal{N}_{e^{-(r+b_*)}}(\varphi C_2)$ for some $[\varphi] \in \Gamma/\Gamma_2$ with $s_\lambda \leq r$. Moreover, by assumption, $\varphi C_2 \in \mathcal{S}$ is contained in a metric sphere with respect to d_o . By (3.5), the radius of this metric sphere is at least $e^{-D_2([\varphi])}/2 \geq e^{-s_\lambda} \geq e^{-r}$. Thus, since X is b_* -diffuse with respect to \mathcal{S} and by (2.21), X is strongly b_* -diffuse with respect to \mathcal{F}_2 and we have that **Bad**(\mathcal{D}_2) is ψ_s -winning by Theorem 2.4.

3.6.6. Verifying (b_*, d_*, n_*) for Case 3. We switch to the hyperbolic ball model and assume that $o = 0$ is the center. Let $\Lambda\Gamma$ be the support of a locally finite Borel measure μ which satisfies a power law with exponent $\tau > 0$. For a subset $M \subset S^n$ and $0 \leq a < b$, consider the truncated cone of M with respect to o ,

$$M(a, b) \equiv \{\gamma_{o,\xi}(t) \in \mathbb{H}^{n+1} : \xi \in M, a \leq t \leq b\}.$$

We show that μ is (ψ_s, f, b_*) -decaying with respect to \mathcal{F}_3 , where $f(b, s) = c \cdot be^{-\tau s}$ for some constant $c > 0$. To this end, let $b > 0$ and $t_* > -\log(R)$ be sufficiently large (which we may assume since $t_0 > 0$ is sufficiently large) and fix a formal ball $\omega = (z, r) \in \Lambda\Gamma \times (t_*, \infty) = \Omega$. Note that for any point $\xi \in B_{d_o}(z, e^{-r} + e^{-(r+b)}) \subset B_{d_o}(z, 2e^{-r})$, $z \in \Lambda\Gamma$, we have $(\xi, z)_o \geq \log(d_o(\xi, \eta)) \geq r - \log(2)$ and hence,

$$d(\gamma_{o,\xi}(r), \gamma_{o,z}(r)) \leq d(\gamma_{o,\xi}(r - \log(2)), \gamma_{o,z}(r - \log(2))) + 2\log(2) \leq \delta + 2\log(2).$$

By convexity, we have that the truncated cone $(B_{d_o}(z, e^{-r} + e^{-(r+b)}))(\max\{r - b, 0\}, r)$ is contained in the $(\delta + 2\log(2))$ -neighborhood of the geodesic segment $\gamma_{o,z}([r - b, r])$. Let $G(z, r, b)$ denote the orbit points of $\Gamma.x$ in this truncated cone and note that there exists a

$d = d(x, \Gamma) > 0$ such that the orbit $\Gamma.x$ is d -separated. Hence, since \mathbb{H}^{n+1} is of constant sectional curvature, it follows by a usual (hyperbolic) volume argument that there exists a constant $C_1 = C_1(d)$ independent of b and r such that

$$|G(z, r, b)| \leq C_1 \cdot (r - (r - b)) = C_1 \cdot b.$$

Note that $\{\psi(x)_\infty : \psi(x) \in G(z, r, b)\} \subset R(r, b)$, and moreover, if the metric ball $B_{d_o}(\psi(x)_\infty, e^{-(r+b)})$ intersects $\psi_s(\omega) = B_{d_o}(z, e^{-r})$ then $\psi(x) \in G(z, r, b)$. Hence,

$$\begin{aligned} \mu(B_{d_o}(z, e^{-r}) \cap \mathcal{N}_{e^{-(r+s)}}(R(r, b))) &\leq \bigcup_{\psi(x) \in G(z, r, b)} \mu(B_{d_o}(\psi(x)_\infty, e^{-(r+s)})) \\ &\leq C_1 b \cdot c_2 e^{-\tau(r+s)} \leq \frac{C_1 c_2}{c_1} b e^{-\tau s} \mu(B_{d_o}(z, e^{-r})) \\ &\equiv f(b, s) \mu(B_{d_o}(z, e^{-r})). \end{aligned}$$

Clearly, for every $n_* \in \mathbb{N}$, $d_* \geq \log(2)$, the function f satisfies $f(n_* b + d_*, b - 2d_*) \leq \tilde{c} < 1$ for all $b > b_* \geq 2d_*$, where $\bar{b}_* = \bar{b}_*(d_*, n_*) \geq 2d_*$ is sufficiently large. Hence $\Lambda\Gamma = \text{supp}(\mu)$ is (b_*, d_*, n_*) -diffuse with respect to \mathcal{F}_3 by Proposition 2.9. We follow that $\mathbf{Bad}(\mathcal{F}_3)$ and $F(\mathbf{Bad}(\mathcal{F}_3))$ are ψ_s -weakly-winning from Theorem 2.4 and Proposition 2.5, for any L -bi-Lipschitz map $F : \Lambda\Gamma \rightarrow F(\Lambda\Gamma)$ with $2L \leq d_*$. Since (2.26) and (MSG1-2) are satisfied, we have from Theorem 2.10 that $\dim(\mathbf{Bad}(\mathcal{F}_3)) = d_\mu(\Lambda\Gamma) = \tau$. The same is true for $F(\mathbf{Bad}(\mathcal{F}_3)) \subset F(\Lambda\Gamma)$.

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